

Standard Model with extra dimensions II: One-loop structure, regularization, renormalization, and asymptotic freedom

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We study the divergent character, at one loop, of the Standard Model with extra dimensions introduced in the first part of this two-paper series. The infinite number of Kaluza-Klein (KK) particles, contained in the theory, can introduce divergences, in addition to those associated with short-distance effects. We introduce a compactification scheme that geometrically recreates the Casimir's effect, so inhomogeneous Epstein's functions arise in loop amplitudes. When the dimensional regularization scheme is introduced, we find that nonstandard divergences arising from the unbounded number of KK particles and divergences associated with short-distance effects are naturally described by the poles of the Epstein's function and the Gamma function, respectively, thus unifying, in this sense, the regularization of both types of divergences. By using a regularized (finite) version of the inhomogeneous Epstein's function, we show that both ultraviolet divergences and nonstandard divergences are consistently removed from loop amplitudes. We show that, at the one-loop level, the ultraviolet divergences of the model are ultimately related solely to Standard Model particles. These facts lead us to a highly predictive theory, which is renormalizable in a broader or modern sense. We present explicit expressions for the Passarino-Veltman scalar functions arising from loops of KK particles. As an application, with the use of covariant gauge-fixing procedures, the one-loop effective action for non-Abelian theories is calculated; the renormalization prescription that allows us to define appropriate counterterms is introduced and the corresponding β function is presented. This β function is finite and negative regardless of the number of extra dimensions being at stake; asymptotic freedom prevails and is made acute in the presence of extra dimensions.

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I. INTRODUCTION

In Ref.[1], from now on referred to as Part I, we presented an effective theory for the Standard Model (SM) with extra dimensions. In this second part, we will explore some technical issues and phenomenological implications at the level of radiative corrections in this theory. In particular, we will focus on the divergent behavior of the model at the one-loop level, for it is at this order that SM observables are firstly influenced by the presence of Kaluza-Klein (KK) modes [2]. It turns out that, besides the ultraviolet divergences already present in the SM, new divergences of this type may be generated by the KK excitations of the standard fields. Furthermore, a new kind of divergences, in principle, emerge due to the presence of an infinite number of KK excitations. In this SM extension, Green's functions can naturally be classified into three categories: standard Green's functions (SGF), defined as those whose external legs belong to SM fields; hybrid Green's functions (HGF), whose external legs correspond to both SM fields and KK excitations; and nonstandard Green's functions (NSGF), whose external legs belong to KK excitations. It turns out that divergences associated with the infinite number of KK excitations could emerge in SGF, which plays a central role in predicting electroweak observables. The existence of this class of divergences, which do not arise from short-distance effects in the four-dimensional spacetime manifold but from the contribution of an infinite number of KK excitations, has been generally accepted in the literature. However, no consensus in treating them has been reached. The divergent structure of the theory depends crucially on how the compactification of the extra dimensions is carried out. Our proposal consists in the introduction of a compactification scheme that geometrically recreates the physical essence of the famous Casimir effect, whose mathematical foundations have been widely studied in the literature. In Part I of the paper, we introduced a compactification scheme that reproduces the same boundary conditions of the Casimir's effect, which consists in assuming the \mathcal{N}^n manifold to be n copies of the orbifold S^1/Z_2 , where a different radius R_i can be taken at each copy. In the orbifold compactification process, the $\varphi_N(x, \bar{x})$ fields governed by the extended $\{\text{ISO}(1, 3 + n), G(\mathcal{M}^d)_{\text{SM}}\}$ groups, which are periodic on the \bar{x} coordinates, are expanded into even or odd

Fourier series, subject to satisfy the Neumann and Dirichlet boundary conditions on the fixed points $\bar{x}_i = 0$ and $\bar{x}_i = \pi R_i$ of each orbifold S^1/Z_2 . The effective field theory that emerges after compactification can be divided into two parts, one of which only involves interactions of canonical dimension less than or equal to four (which in particular contains the SM Lagrangian), and another containing nonrenormalizable interactions of canonical dimension higher than four (see Part I). So, this theory is nonrenormalizable in the Dyson's sense. However, as we will argue below, this effective theory is predictive since it is renormalizable in a broader or modern sense [3–8].

In the study of the one-loop structure of this version of the SM with extra dimensions the Epstein zeta function naturally emerges, which is a generalization to higher dimensions of the famous Riemann zeta function [9–12]. Playing an active role in pure mathematical discussions, especially within number theory [13], the Riemann zeta function and its extensions are very interesting objects in their own right. The systematic use in physics of zeta-regularization methods dates from the 1970's, with seminal works by J. S. Dowker and R. Critchley [14], within the context of effective field theories, and by S. W. Hawking [15], in integrals on curved spacetimes. Applications of zeta functions can also be spotted in quantum gravity models and cosmology [16], string theory [17], and crystallography [18]. For more details on applications of zeta functions and some mathematical insight, the reader is referred to the books [19, 20]. An outstanding application of zeta functions is present in the calculation of the physical vacuum energy, or Casimir energy, of a quantized field in presence of external boundary configurations, from which the Casimir force between perfect conducting plates is derived [21]. The computation of the vacuum energy for scalar fields at different temperature limits may be solved by the use of the Epstein zeta function [22, 23]. As we will see below, in our case the SGFs depend crucially on the inhomogeneous Epstein function, which has already been the object of attention in the literature [23–25]. In particular, we will make extensive use of a regularized (finite) version of this function that has been derived by Elizalde-Romeo [24] following the scheme introduced by Weldon in Ref. [25]. In this work, we will follow closely the techniques developed by these authors. As a direct consequence of using this method, we will show that, at the one-loop level, the ultraviolet divergences present in SGFs emerge only from the SM fields. Thereby, one of the main goals of the present work is to use the Epstein zeta function as the main tool in a regularization scheme to make sense of the physical amplitudes within the framework of the extra-dimensional SM (EDSM) proposed in Part I. At first sight, this regularization scheme could seem rather artificial if introduced without an appropriate physical argumentation. Indeed, both physical and technical motivations are necessary due to the counterintuitive behavior of the zeta functions. We will show that these issues can be successfully addressed by following a line strongly linked to dimensional regularization [27].

We will show that this SM extension to extra dimensions, equipped with regularized Epstein's functions, is much less complicated than one might expect, because it does not have any type of divergences associated with the KK particles. These facts will allow us to argue that these effective theories are predictive since they are renormalizable in a broader or modern sense [3–8]. We will show that this is so by presenting a comprehensive study of asymptotic freedom. It is a well-known fact that the coupling constant in Yang-Mills theories becomes stronger in the large-distances regime and weaker at short distances. This noteworthy phenomenon, known as asymptotic freedom, plays an essential role in the formulation of the quantum field theory that describes the strong interactions. Asymptotic freedom requires a negative β function. It has been well established that gauge fields contribute negatively to this function whereas matter fields do it positively. In strong interactions, gluons dominate over quark species. Our purpose in this part of the paper is to investigate if this delicate balance remains in the presence of extra dimensions. We will illustrate how renormalization in a modern sense can be implemented in this class of effective theories. We decided to explore asymptotic freedom in our model due to its importance in the SM and, in particular, in quantum Yang-Mills theories.

The rest of the paper has been organized as follows. In Sec. II, we discuss the divergent structure of the EDSM. In particular, we introduce a regularization scheme to handle the divergent structure of the various physical amplitudes in radiative corrections. Sec. III is devoted to present a comprehensive study of asymptotic freedom. In Sec. IV, our conclusions are presented.

II. ONE-LOOP STRUCTURE

This section is devoted to study the divergent structure, at the one-loop level, of the EDSM discussed in Part I of this work. For reasons that will become apparent later, we write the corresponding effective Lagrangian as a sum of three effective Lagrangians,

$$\mathcal{L}_{\text{eff}}^{(0)} = \mathcal{L}_{\text{SM}}^{(0)} + \mathcal{L}_{\text{d}>4}^{(0)}, \quad (\text{II.1})$$

$$\mathcal{L}_{\text{eff}}^{(0)(m)} = \mathcal{L}_{\text{d}=4}^{(0)(m)} + \mathcal{L}_{\text{d}>4}^{(0)(m)}, \quad (\text{II.2})$$

$$\mathcal{L}_{\text{eff}}^{(m)} = \mathcal{L}_{\text{d}=4}^{(m)} + \mathcal{L}_{\text{d}>4}^{(m)}, \quad (\text{II.3})$$

where the first terms of these Lagrangians, which were explicitly derived in Sec. IV of Part I, emerge from compactification of the $(4+n)$ -dimensional version of the SM. All the interactions appearing in these Lagrangians have a renormalizable structure in the Dyson's sense, which is indicated by the subscript $\mathbf{d} = 4$. The $\mathcal{L}_{\text{SM}}^{(0)}$ Lagrangian represents the SM, $\mathcal{L}_{\mathbf{d}=4}^{(0)(\underline{m})}$ contains interactions among SM fields (zero modes) and KK excitations, and $\mathcal{L}_{\mathbf{d}=4}^{(\underline{m})}$ involves interactions only among KK excitations. The $\mathcal{L}_{\mathbf{d}=4}^{(0)(\underline{m})}$ and $\mathcal{L}_{\mathbf{d}=4}^{(\underline{m})}$ Lagrangians are given by infinite series in the Fourier indices (\underline{m}) . Contributions to physical amplitudes coming from this sector of the EDSM will depend on the Fermi, v , and compactification, R^{-1} , scales.

On the other hand, the last terms in Eqs. (II.1), (II.2), and (II.3) come from compactification of the second term in the effective Lagrangian given by Eq. (II.2) of Part I. All these terms involve interactions of canonical dimension higher than four. All the interactions that respect the $\{\text{ISO}(1,3), G(\mathcal{M}^4)_{\text{SM}}\}$ symmetries must be included because there is no criterion to exclude them, so these Lagrangians are given indeed by an infinite sum over all allowed canonical dimensions. The $\mathcal{L}_{\mathbf{d}>4}^{(0)(\underline{m})}$ and $\mathcal{L}_{\mathbf{d}>4}^{(\underline{m})}$ Lagrangians include, in addition, infinite sums over Fourier indices. It should be noted that the expression given by Eq. (II.1) corresponds to the most general Lagrangian that extends the SM in a model independent fashion. Here, we will refer to it as a conventional effective theory. Observe the great similitude among these three Lagrangians. From now on, we will refer to $\mathcal{L}_{\text{eff}}^{(0)}$ as a conventional effective Lagrangian, whereas $\mathcal{L}_{\text{eff}}^{(0)(\underline{m})}$ and $\mathcal{L}_{\text{eff}}^{(\underline{m})}$ will be called nonconventional effective Lagrangians. Contributions from those sectors with $\mathbf{d} > 4$ are suppressed by inverse powers of the Λ scale, which expected to be well above of the compactification scale R^{-1} .

A. Review of ζ functions

In the EDSM, the one-loop extra-dimensions contributions to SGFs would involve sums of the way

$$\sum_{(\underline{m})} \frac{1}{(\underline{m}^2 + c^2)^s}, \quad \underline{m}^2 = \underline{m}_1^2 + \cdots + \underline{m}_n^2, \quad s \in \mathbb{Z}, \quad (\text{II.4})$$

where c is a constant. In the expression given in Eq. (II.4), the symbol $\sum_{(\underline{m})}$ involves simple sums, and higher nested series. A special case of this class of sums corresponds to $c = 0$, in which some of the simple series are convergent, as those given by the Dirichlet series

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s}, \quad \text{Re}(s) > 1. \quad (\text{II.5})$$

For instance, $\zeta(2) = \pi^2/6$. In contrast, since Eq. (II.4) also allows for negative integer values of s , we will encounter divergent series as well, *e.g.* $\sum_{m=1}^{\infty} \frac{1}{m^s}$, for $\text{Re}(s) < 1$. There is, however, a technique for $\text{Re}(s) < 0$ using [28]

$$\zeta(s) = \frac{\pi^{s-\frac{1}{2}} \Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})} \zeta(1-s). \quad (\text{II.6})$$

This reflection formula allows us to define values of the ζ functions on new domains. From this expression, the counterintuitive particular value

$$\zeta(-1) = \sum_{m=1}^{\infty} m = -\frac{1}{12}, \quad (\text{II.7})$$

is obtained. Zeros of the ζ function at negative even integers $s = -2, -4, \dots, -2n$, also called trivial zeros, arise from the singularity of $\Gamma(s/2)$. Although $\Gamma(s/2)$ is also singular at $s = 0$, it is compensated by the singularity at $\zeta(1)$ -the harmonic series- giving a nonzero finite value for $\zeta(0)$, namely $\zeta(0) = -1/2$. The values of the ζ function for points that lie in the so-called critical strip, defined by $0 \leq \text{Re}(s) \leq 1$, arise from analytically continuing towards $\text{Re}(s) = 0$, through the following identity [28]:

$$\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m^s} = (1 - 2^{1-s}) \zeta(s), \quad \text{Re}(s) > 1. \quad (\text{II.8})$$

It turns out that the alternating series on the left-hand side converges for all $\text{Re}(s) > 0$. Thereby Eq. (II.8) provides a formula that coincides with the values of the ζ function for $\text{Re}(s) > 1$ and becomes an analytic continuation of the

ζ function throughout the critical strip. Then the values on the critical strip are covered by

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m^s}, \quad 0 \leq \Re(s) \leq 1. \quad (\text{II.9})$$

In conclusion, the values for the series $\sum_{m=1}^{\infty} \frac{1}{m^s}$ can be defined on the critical strip by Eq. (II.9), to the right of this domain by (II.5), and to the left of it by (II.8). A pole is found at $s = 1$. The Riemann hypothesis states that all non-trivial zeros of ζ lie on the critical line consisting of the complex numbers $1/2 + it$, which are not of our interest since $s \in \mathbb{R}$ in our applications.

A generalization of the Riemann ζ function is the Hurwitz ζ function [29], defined by

$$\zeta_{\text{H}}(s, a) = \sum_{m=0}^{\infty} \frac{1}{(m+a)^s}, \quad s \in \mathbb{C}, \quad \Re(s) > 1, \quad a \neq 0, -1, -2, \dots \quad (\text{II.10})$$

Notice that for $a = 1$, $\zeta_{\text{H}}(s, a)$ reduces to the Riemann zeta function $\zeta(s)$. Its analytical continuation to the complete complex plane, except for $s = 1$, is given by

$$\zeta(s, a) = \frac{1}{s-1} \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{\infty} (-1)^k \frac{n!}{k!(n-k)!} (a+k)^{1-s}. \quad (\text{II.11})$$

As we mentioned before, despite the control we would have for simple series, our task should not end at this point as we also have double, triple, and, in general, n -tuple nested series. It is remarkable that each of these discrete summations correspond to a particular case of one of various so-called Epstein zeta functions [10, 19], which are generalizations to higher dimensions of the Riemann zeta function. In particular, it is interesting the analytical continuation of [30]

$$Z(s; Q) = \sum_{0 \neq x \in \mathbb{Z}^n} \frac{1}{(x^T Q x)^s}, \quad \Re(s) > \frac{n}{2}, \quad (\text{II.12})$$

where $s \in \mathbb{C}$, Q is a positive $n \times n$ symmetric matrix, x is a column vector and x^T is its transpose, so that we have the quadratic form $x^T Q x = \sum_{i,j=1}^n Q_{ij} x_i x_j$. This function has an analytic continuation to the complete complex plane, except for a simple pole at $s = n/2$. The functional equation or reflection relation satisfied by this function is Riemann-like [c.f. (II.8)]:

$$Z(s; Q) = (\det Q)^{-\frac{1}{2}} \frac{\pi^{2s-\frac{n}{2}} \Gamma(n/2 - s)}{\Gamma(s)} Z(n/2 - s; Q^{-1}). \quad (\text{II.13})$$

As in the Riemann zeta function case, trivial zeros given by $s = -m$, with $m \in \mathbb{N}$, arise from the singularity of $\Gamma(s)$. Nontrivial zeros are complex numbers denoted by $\rho = \beta + i\gamma$. As in the Riemann zeta function case, not much is known about their distribution [31]. In the especial case $Q = I_n$, with I_n the $n \times n$ identity matrix, we express the above functional equation as

$$Z(s; I_n) = \frac{\pi^{2s-\frac{n}{2}} \Gamma(n/2 - s)}{\Gamma(s)} Z(n/2 - s; I_n). \quad (\text{II.14})$$

This is the homogeneous Epstein zeta function, which is denoted by

$$Z_l(s) = \sum_{\substack{m_1, \dots, m_l = -\infty \\ m_1, \dots, m_l \neq 0}}^{+\infty} \frac{1}{(m_1^2 + \dots + m_l^2)^s}. \quad (\text{II.15})$$

On the other hand, the inhomogeneous Epstein zeta function is defined by

$$E_l^{c^2}(s) = \sum_{m_1, \dots, m_l=1}^{+\infty} \frac{1}{(m_1^2 + \dots + m_l^2 + c^2)^s}, \quad (\text{II.16})$$

where c is a constant. This function has simple poles at $s = \frac{l}{2}, \frac{l-1}{2}, \dots$, except for $s = 0$ or negative integers [24, 32, 33]. A special case corresponds to $c = 0$,

$$E_l(s) = \sum_{m_1, \dots, m_l=1}^{+\infty} \frac{1}{(m_1^2 + \dots + m_l^2)^s}, \quad (\text{II.17})$$

which is singular at $s = \frac{l}{2}, \frac{l-1}{2}, \dots, \frac{1}{2}$ and has trivial zeros given by $s = -1, -2, -3, \dots$. This function is related to the homogeneous Epstein zeta function through the following relation [34]:

$$E_d(s) = \sum_{m=1}^d \frac{(-1)^{m+d}}{2^d} \frac{d!}{m!(d-m)!} Z_m(s). \quad (\text{II.18})$$

The homogeneous Epstein zeta function $Z_l(s)$ can be expressed as products of unidimensional sums [35]. The algorithm to perform the dimensional reduction of $Z_l(s)$, with emphasis in the odd-dimensional case, which is the most intricate one, has been done in [34].

In practical one-loop calculations, we have to deal with multidimensional sums of the type given by Eq. (II.6) of Part I. As we will see below, such type of sums can be expressed in terms of inhomogeneous Epstein functions as

$$\sum_{(\underline{m})} \frac{1}{(\underline{m}^2 + c^2)^s} = \sum_{l=1}^n \frac{n!}{l!(n-l)!} E_l^{c^2}(s). \quad (\text{II.19})$$

B. Divergent structure of Green's functions at the one-loop level

Our effective theory for the EDSM has the main ingredients of conventional field theories, but it has further interesting features. In first place, this effective theory is made of $\{\text{SO}(1, 3), G(\mathcal{M}^4)\}$ -invariant interactions constructed out not only with the SM fields (the zero mode fields), but also with an infinite number of KK fields, which have well-defined laws of transformation under these groups. As we already commented, the effective Lagrangian of the theory has a sector that involves interactions among SM fields and KK excitations that are well behaved at the loop level because they are renormalizable in the Dyson's sense (see Eq.(II.2)). One further ingredient of this part of the EDSM is the fact that the nonstandard fields have masses determined by products of Fourier indices with the compactification scale R^{-1} (see Sec. II of Part I), which is expected to be quite above the Fermi scale $v = 246$ GeV. Of course, we can build conventional effective Lagrangians for some extensions of the SM, as, for example, the SM with extended scalar sectors¹. The striking difference with our case is the presence of an infinite number of fields, whose collective contribution may eventually lead to divergences. So, the presence of an unlimited number of fields can give rise to two types of divergences in radiative corrections to electroweak observables: besides those divergences that are associated with short-distance effects in quantum field theories, in this class of effective theories divergences can also occur because one must consider the virtual contributions of an infinite number of particles. This is the case of the one-loop contribution of KK excitations to SGF. This already suggests that, in general, we must deal to two kinds of divergences in the EDSM. In order to distinguish this type of divergences from standard divergences (SD) associated with short-distance effects, from now on we will refer to them as nonstandard divergences (NSD). However, as already emphasized in the Introduction, the structure of NSDs depends crucially of the geometry introduced in the compactification of the extra manifold \mathcal{N}^n . Motivated by the well known results on the Casimir's effect, we have introduced a compactification scheme which resembles the main mathematical features of this problem (see Part I).

In general, at the one-loop level, the contribution to a SGF, $\Gamma_N^{(0)}$, will involve discrete infinite sums (series) over all allowed Fourier modes (sums over all contributions of KK particles circulating in the loop), besides the usual continuum sums (integrals) over momenta involved in loops, that is,

$$\Gamma_N^{(0)} \sim \sum_{(\underline{m})} \int \frac{d^4 k}{(2\pi)^4}. \quad (\text{II.20})$$

Both continuum and discrete sums may lead to divergencies. As it is usual in conventional renormalizable theories, the former type of divergences are handled by means of dimensional regularization. The main goal of this section is

¹ See, for instance, Ref. [38].

to present a regularization scheme that allows us to handle the divergent discrete sums. Once regularized the SGFs, we will be in position of implementing a renormalization program in a modern sense [3–8] that allows us to remove consistently the divergences and thus to define finite physical amplitudes.

As already commented in the Introduction, in the EDSM three classes of Green’s functions can be defined, which we called SGFs, HGFs, and NSGFs. A comprehensive study about the one-loop structure and physical meaning of both HGFs and NSGFs will be presented elsewhere [36]. In this work we focus on the SGFs.

The SGFs play a central role in phenomenology because they determine the impact of extra dimensions on SM observables. It is worthwhile noticing that no vertex in the $\mathcal{L}_{d=4}^{(0)(m)}$ Lagrangian contains only one KK excitation, but every vertex with excited modes contains at least two KK excited fields of the same type. This is a consequence of momentum conservation at the higher-dimensional level. The immediate implication is that decays of KK excitations into zero modes are forbidden at tree level. One-loop SGFs only receive contributions from the $\mathcal{L}_{\text{eff}}^{(0)}$ and the $\mathcal{L}_{\text{eff}}^{(0)(m)}$ effective Lagrangians. Contributions from the $\mathcal{L}_{\text{eff}}^{(m)}$ Lagrangian to this type of Green’s functions first arise at the two-loop level. At one loop, if the SGFs contain KK excited modes they must be circulating around the corresponding loops. This amounts to the presence of discrete infinite sums (series) over all allowed Fourier modes, besides the usual continuum sums (integrals) over momenta involved within loops. Both continuum and discrete sums may lead to divergences, namely, SDs and NSDs, respectively. As already commented, the former type of divergences will be handled by means of dimensional regularization, whereas the latter will be isolated through the inhomogeneous Epstein function.

Let $\Gamma_N^{(0)\mu\nu\cdots} = \Gamma_N^{(0)} T_{\mu\nu\cdots}^{\rho\sigma\cdots}$ be a tensorial vertex SGF, being $T_{\mu\nu\cdots}^{\rho\sigma\cdots}$ a Lorentz tensor of a given rank, whose specific form is irrelevant for the forthcoming discussion. The number of one-loop diagrams contributing to $\Gamma_N^{(0)}$ depends, in general, on the type of particles circulating in the loop, as well as on the gauge-fixing procedure used. Taking into account that there are no tree-level contributions from KK excitations to this type of Green’s functions, one can write, up to one-loop,

$$\Gamma_N^{(0)} = \Gamma_N^{\text{tree}} \left(\mathcal{L}_{\text{eff}}^{(0)} \right) + \Gamma_N^{\text{loop}} \left(\mathcal{L}_{\text{eff}}^{(0)} \right) + \Gamma_N^{\text{loop}} \left(\mathcal{L}_{\text{eff}}^{(0)(m)} \right), \quad (\text{II.21})$$

where

$$\Gamma_N^{\text{tree}} \left(\mathcal{L}_{\text{eff}}^{(0)} \right) = \Gamma_N^{\text{tree}} \left(\mathcal{L}_{\text{SM}}^{(0)} \right) + \Gamma_N^{\text{tree}} \left(\mathcal{L}_{d>4}^{(0)} \right), \quad (\text{II.22a})$$

$$\Gamma_N^{\text{loop}} \left(\mathcal{L}_{\text{eff}}^{(0)} \right) = \Gamma_N^{\text{loop}} \left(\mathcal{L}_{\text{SM}}^{(0)} \right) + \Gamma_N^{\text{loop}} \left(\mathcal{L}_{d>4}^{(0)} \right), \quad (\text{II.22b})$$

$$\Gamma_N^{\text{loop}} \left(\mathcal{L}_{\text{eff}}^{(0)(m)} \right) = \Gamma_N^{\text{loop}} \left(\mathcal{L}_{d=4}^{(0)(m)} \right) + \Gamma_N^{\text{loop}} \left(\mathcal{L}_{d>4}^{(0)(m)} \right). \quad (\text{II.22c})$$

In the above expressions, Eqs. (II.22a) and (II.22b) are the contributions to $\Gamma_N^{(0)}$ induced by a conventional effective theory $\mathcal{L}_{\text{eff}}^{(0)}$ at the tree level and one-loop level, respectively; whereas Eq. (II.22c) is the corresponding contribution induced by the nonconventional effective theory $\mathcal{L}_{\text{eff}}^{(0)(m)}$. Effective contributions at the tree level given by $\Gamma_N^{\text{tree}} \left(\mathcal{L}_{d>4}^{(0)} \right)$ can emerge for $N \leq 4$ due to the shift generated by the Higgs mechanism. The presence of these tree-level contributions for $N > 4$ means that the counterterm needed to renormalize the theory in a modern sense [3–8] is already available in the conventional effective $\mathcal{L}_{\text{eff}}^{(0)}$ Lagrangian. The one-loop $\Gamma_N^{\text{loop}} \left(\mathcal{L}_{\text{eff}}^{(0)} \right)$ contribution only can be affected by SDs. The first term in Eq. (II.22b) corresponds to the SM contribution, so that divergences only can arise for $N \leq 4$. On the other hand, the second term in this expression corresponds to contributions induced by the $\mathcal{L}_{d>4}^{(0)}$ Lagrangian. Due to the shift generated by the Higgs mechanism, this effective Lagrangian can induce both renormalizable and nonrenormalizable interactions, so SDs can or cannot arise depending on the type of vertices considered. Typical contributions of this class are given by insertions, in the SM diagrams defining the SGF in consideration, of vertices that depend on the unknown scale Λ . Since this class of effective theories has already been studied in the literature [3–8, 37], we will deal with this only circumstantially. So, from now on, we will focus on the $\Gamma_N^{\text{loop}} \left(\mathcal{L}_{\text{eff}}^{(0)(m)} \right)$ contribution to $\Gamma_N^{(0)}$.

The $\Gamma_N^{\text{loop}} \left(\mathcal{L}_{d=4}^{(0)(m)} \right)$ contribution, which is induced by renormalizable couplings among SM fields and their KK excitations, can have both SDs and NSDs if $N \leq 4$, but only the latter type of divergence may arise for $N > 4$. The absence of SDs for $N > 4$ is obvious from the fact that only renormalizable interactions in the Dyson’s sense are present. In regard to the $\Gamma_N^{\text{loop}} \left(\mathcal{L}_{d>4}^{(0)(m)} \right)$ contribution, it emerges from nonrenormalizable vertices that involve both SM fields and their KK excitations. In this case, the contributions are proportional to the Λ scale and both

SDs and NSDs may arise. However, we will show that in our framework, which incorporates regularized (finite) Epstein functions, NSDs arising in combination with SDs conspire to produce a finite value. As in the case of the $\Gamma_N^{\text{loop}} \left(\mathcal{L}_{\mathbf{d}=4}^{(0)(\underline{m})} \right)$ contribution, NSDs can, in general, appear due to infinite sums which do not involve SDs, but they are not present in the framework used in this work.

From the above discussion it is clear that we have to deal with a theory that is not renormalizable in the Dyson's sense. However, effective field theories given by the general $\mathcal{L}_{\text{eff}}^{(0)}$ Lagrangian are predictive because they are renormalizable in a modern sense [3–8]. Crucial to these theories is the separation of the physical phenomena that can be explored at accessible energies from those which only show up at much higher energies. According to renormalizability in a modern sense, one can carry out radiative corrections using an effective Lagrangian, which, by definition, includes interactions that are not renormalizable in the Dyson's sense. New types of infinities can arise, but this does not constitute a serious problem, as the counterterms needed to remove them are already present in the effective Lagrangian. Such divergences simply renormalize the bare coupling constants that multiply interactions of canonical dimension higher than four. This technique has been applied by many authors to estimate corrections to electroweak observables induced by insertions of nonrenormalizable vertices in loop graphs [37, 38].

C. Kaluza-Klein divergences

We now proceed to study the divergent structure of the $\Gamma_N^{\text{loop}} \left(\mathcal{L}_{\text{eff}}^{(0)(\underline{m})} \right)$ contribution. In radiative corrections at the one-loop level, typical tensor amplitudes of arbitrary range can be reduced to expressions given in terms of the A_0 , B_0 , C_0 , and D_0 scalar functions through the Passarino-Veltman algorithm [39], which in turn can be expressed in terms of elementary functions using an appropriate Feynman parametrization. An important objective of this section is to extend this algorithm for the case when there are extra dimensions. Consider a N -point scalar function F_N that is induced by the $\mathcal{L}_{\mathbf{d}=4}^{(0)(\underline{m})}$ or $\mathcal{L}_{\mathbf{d}>4}^{(0)(\underline{m})}$ Lagrangians at the one-loop level and which is defined as follows:

$$F_N = \sum_{(\underline{m})} \frac{1}{i\pi^2} \int d^4k \frac{1}{\left[k^2 - m_{\varphi_0(\underline{m})}^2 \right] \left[(k + p_1)^2 - m_{\varphi_1(\underline{m})}^2 \right] \cdots \left[(k + p_{N-1})^2 - m_{\varphi_{N-1}(\underline{m})}^2 \right]}. \quad (\text{II.23})$$

After a Feynman parametrization, the above expression becomes

$$F_N = \frac{1}{i\pi^2} \Gamma(N) \int_0^1 dx_1 \cdots dx_N \delta \left(\sum_{i=1}^N x_i - 1 \right) \sum_{(\underline{m})} \int d^4k \frac{1}{\left(k^2 - \Delta_{(\underline{m})}^2 \right)^N}, \quad (\text{II.24})$$

where $\Delta_{(\underline{m})}^2$ is a quadratic function on the x_i variables, external momenta, and KK internal masses, which, by virtue of Eq.(II.11) of Part I, can be written in general as follows:

$$\Delta_{(\underline{m})}^2 = m_{(\underline{m})}^2 + \Delta_{(\underline{0})}^2, \quad (\text{II.25})$$

where the $\Delta_{(\underline{0})}^2$ function is the expression that would result from substituting in Eq.(II.23) all the KK masses by their SM counterpart. This function can be written as follows:

$$\Delta_{(\underline{0})}^2 = m_{\varphi_0(\underline{0})}^2 + \sum_{i,j=1}^{N-1} p_i p_j x_i x_j - \sum_{i=1}^{N-1} \left(p_i^2 + m_{\varphi_0(\underline{0})}^2 - m_{\varphi_i(\underline{0})}^2 \right) x_i. \quad (\text{II.26})$$

Clearly, the integral on k in (II.24) diverges for $N \leq 2$, so SDs are induced in these cases. In addition, NSDs can arise from the diverse sums nested in the $\sum_{(\underline{m})}$ symbol. As it is usual, SDs are handled through the dimensional regularization scheme. As we emphasized in the Introduction, one important goal of this work is to introduce a regularization scheme to deal with the NSDs. Our approach consists in regularizing simultaneously both SDs and NSDs, since, after all, they have the same origin in the sense that the sums in consideration involve the magnitudes k and $p_{\vec{\mu}}^{(\underline{m})}$, which are linked to the manifolds \mathcal{M}^4 and \mathcal{N}^n through Fourier transform (k_{μ}) and Fourier series ($p_{\vec{\mu}}^{(\underline{m})}$). The main idea behind this is to express the $\sum_{(\underline{m})}$ sums in terms of Epstein zeta functions, which are defined in the complex plane through analytical continuation of less general series defined on some region of the real axis.

As usual, we promote the ordinary four spacetime dimensions to D dimensions. Once this is done, Eq.(II.24) becomes

$$\begin{aligned}
F_N &= \frac{1}{i\pi^2} (\mu^2)^{(2-\frac{D}{2})} \Gamma(N) I_N \sum_{(\underline{m})} \int d^D k \frac{1}{(k^2 - \Delta_{(\underline{m})}^2)^N} \\
&= (-1)^N \left(\frac{1}{4\pi\mu^2} \right)^{(N-2)} I_N \sum_{(\underline{m})} \Gamma\left(N - \frac{D}{2}\right) \left(\frac{\Delta_{(\underline{m})}^2}{4\pi\mu^2} \right)^{-(N-\frac{D}{2})} \\
&= (-1)^N (4\pi\mu^2)^{(2-\frac{D}{2})} (R^{-2})^{(\frac{D}{2}-N)} I_N \sum_{(\underline{m})} \Gamma\left(N - \frac{D}{2}\right) (\underline{m}^2 + c_N^2)^{-(N-\frac{D}{2})}, \tag{II.27}
\end{aligned}$$

where we have introduced the short-hand notation

$$I_N = \int_0^1 dx_1 \cdots dx_N \delta\left(\sum_{i=1}^N x_i - 1\right). \tag{II.28}$$

In addition, in order to simplify the analysis, we have assumed equal radii R for all the orbifolds S^1/Z_2 , so we can write $m_{(\underline{m})}^2 = R^{-2} \underline{m}^2$, with \underline{m}^2 any admissible combination of Fourier indices (see Eq.(II.37) of Part I). In addition, $c_N^2 = \Delta_{(0)}^2/R^{-2}$ and μ is the scale associated with the dimensional regularization scheme.

The crucial point in expression (II.27) is that in order to implement the $D \rightarrow 4$ limit in the Γ function, D must be complex. But this is precisely the link with the dimensional regularization scheme that we need to introduce in the above expression the Epstein functions in order to regularize the NSDs. Using the identity given by Eq.(II.19), expression (II.27) becomes

$$F_N = (-1)^N (4\pi\mu^2)^{(2-\frac{D}{2})} (R^{-2})^{(\frac{D}{2}-N)} I_N \sum_{l=1}^n \binom{n}{l} E_l^{c_N^2} \left(N - \frac{D}{2}\right) \Gamma\left(N - \frac{D}{2}\right). \tag{II.29}$$

This is a remarkable result because it shows us that both the Gamma and Epstein functions are defined on the complex plane. As it is well known, the singularities of the Gamma function occur at $N - \frac{D}{2} = 0, -1, -2, \dots$, whereas the inhomogeneous Epstein's function is singular at $N - \frac{D}{2} = \frac{l}{2}, \frac{l-1}{2}, \dots, \frac{1}{2}$. *Thereby, SDs and NSDs emerge as the poles of the Gamma and Epstein functions, respectively.* It is important to note that inhomogeneous Epstein functions naturally arise in the Casimir effect. Their presence in our case should not surprise us, since our compactification process geometrically recreates the boundary conditions of the Casimir effect. Motivated by these facts, we will introduce in our model a regularized (finite) version of the Epstein function, which has already been studied in the literature [24, 25] in other contexts. Following Ref. [24], we perform a binomial expansion of each term of the multiple sums,

$$\begin{aligned}
E_l^{c_N^2} \left(N - \frac{D}{2}\right) &= \sum_{m_1, \dots, m_l=1}^{\infty} (m_1^2 + \dots + m_l^2 + c_N^2)^{-(N-\frac{D}{2})} \\
&= \sum_{m_1, \dots, m_l=1}^{\infty} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(N - \frac{D}{2} + k)}{k! \Gamma(N - \frac{D}{2})} (m_1^2 + \dots + m_l^2)^{-(N-\frac{D}{2}+k)} c_N^{2k}, \tag{II.30}
\end{aligned}$$

which is valid for $c_N^2 \leq 1$. This is the situation in most cases, as $c_N^2 \sim \frac{p^2}{R^{-2}}$, with $\sqrt{p^2}$ the scale of the process, which is expected to be well below the compactification scale R^{-1} . One important exception corresponds to the case of short distance effects, in which $p^2 \gg R^{-2}$, but this special case will be analyzed separately in connection with the phenomenon of asymptotic freedom in Sec.III. On the other hand, if $N - \frac{D}{2}$ was larger than $\frac{l}{2}$, one could commute the sums to obtain

$$E_l^{c_N^2} \left(N - \frac{D}{2}\right) = \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(N - \frac{D}{2} + k)}{k! \Gamma(N - \frac{D}{2})} E_l \left(N - \frac{D}{2} + k\right) c_N^{2k}, \quad N - \frac{D}{2} > \frac{l}{2}, \tag{II.31}$$

with $E_l(N - \frac{D}{2} + k)$ the homogenous Epstein function. However, the commutation of the sums is not acceptable if poles are present. In order to obtain an expression valid for $N - \frac{D}{2} < \frac{l}{2}, \frac{l-1}{2}, \dots, \frac{1}{2}$, we follow the Weldon's

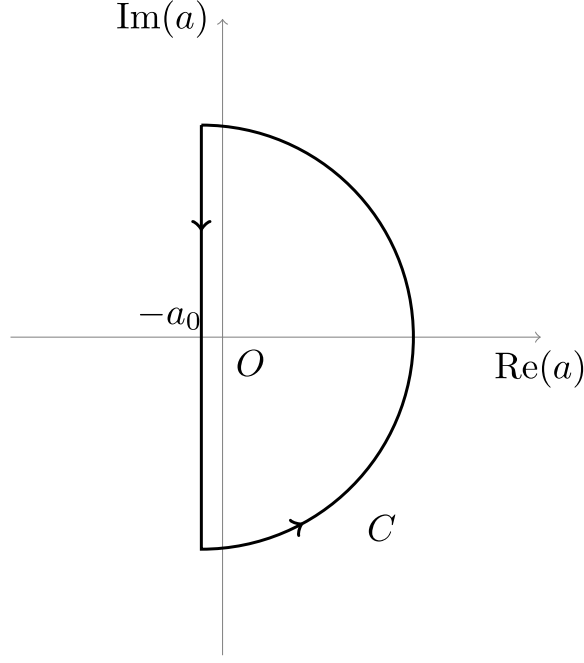


FIG. 1: Contour that defines the integral (II.32).

method [25], which consists in putting the sum over k as a contour integral,

$$E_l^{c_N^2} \left(N - \frac{D}{2} \right) = \sum_{m_1, \dots, m_l=1}^{\infty} \frac{1}{2i} \int_C da \frac{\Gamma(N - \frac{D}{2} + a)}{\Gamma(N - \frac{D}{2} + 1) \Gamma(N - \frac{D}{2})} c_N^{2a} (m_1^2 + \dots + m_l^2)^{-(N - \frac{D}{2} + a)} \csc(\pi a), \quad (\text{II.32})$$

where C is a circuit in the complex plane shown in Fig.1, which in the limit when the radii of the semicircle tends to infinite, encloses all nonnegative integers. In this limit, there is no contribution from the curved part of the contour, as the integrand vanishes on these points. Then, the integral on C can be replaced by the integral on the straight line $\text{Re}(a) = -a_0$, which in turns allows us to move the l_i sums under the integral symbol, leading to an expression given in terms of the homogeneous Epstein function:

$$E_l^{c_N^2} \left(N - \frac{D}{2} \right) = \frac{1}{2i} \int_{-a_0-i\infty}^{-a_0+i\infty} da \frac{\Gamma(N - \frac{D}{2} + a)}{\Gamma(N - \frac{D}{2} + 1) \Gamma(N - \frac{D}{2})} c_N^{2a} E_l \left(N - \frac{D}{2} + a \right) \csc(\pi a). \quad (\text{II.33})$$

To complete the Weldon's method [25], we need to come back to the original circuit C of Fig.1, which now closes all the poles of the integrand. As it has been noted in Refs.[24, 26], the new integrand fails to be zero on the curved part of the circuit C , but this contribution can be neglected if c_N^2 is small enough, which is actually our case. Excluding this term, we can evaluate the above integral by applying the residue theorem to the contour integral:

$$E_l^{c_N^2} \left(N - \frac{D}{2} \right) = \frac{1}{2\pi i} \int_C da \frac{\Gamma(N - \frac{D}{2} + a)}{\Gamma(N - \frac{D}{2} + 1) \Gamma(N - \frac{D}{2})} c_N^{2a} E_l \left(N - \frac{D}{2} + a \right) \pi \csc(\pi a). \quad (\text{II.34})$$

A complete study of this integral has been given in Ref. [24]. Here we limit ourselves to highlighting the main characteristics of this important result. The residues come from three sources: (1) poles of first order of $\csc(\pi a)$ for a a natural integer, (2) poles of first order of $E_l(N - \frac{D}{2} + a)$ for $N - \frac{D}{2} + a = \frac{l}{2}, \frac{l-1}{2}, \dots, \frac{1}{2}$, and (3) poles of first order of $\Gamma(N - \frac{D}{2} + a)$ at $N - \frac{D}{2} = -a$. Note that the other poles of $\Gamma(N - \frac{D}{2} + a)$ at $N - \frac{D}{2} + a = -1, -2, \dots$ are not considered because the product $\Gamma(N - \frac{D}{2} + a) E_l(N - \frac{D}{2} + a)$ is finite at these points, as they are trivial zeros of the homogeneous Epstein function.

One important result of the Weldon's method is that the regularized (finite) Epstein function derived from Eq.(II.34) has trivial zeros at $N - \frac{D}{2} = 0, -1, -2, \dots$ [24]. We now explore the implications of this fact in radiative corrections. When the regularized Epstein function (II.34) is used in the amplitude for the one-loop scalar function (II.29), we have the remarkable fact that

$$\lim_{N - \frac{D}{2} \rightarrow 0, -1, -2, \dots} \Gamma(N - \frac{D}{2}) E_l^{c_N^2} \left(N - \frac{D}{2} \right) = \text{finite}, \quad (\text{II.35})$$

where the $D \rightarrow 4$ limit is implied. This is a surprising result because it tells us that the Weldon's method not only allows us to remove the NSDs but also the short distance effects (SDs) coming from KK particles. Since we did not assume anything concerning the structure of the vertices involved in the loop diagrams, our above results are valid in general. This means that, at the one-loop level, the ultraviolet divergences (SDs) generated by the effective $\mathcal{L}_{\text{eff}}^{(0)(m)}$ Lagrangian are removed simultaneously with the NSDs by the Weldon's method. Note that divergences beyond the quadratic ones can eventually emerge as a consequence of the nonrenormalizable nature of the effective theory, but they always will disappear because, as already mentioned, the regularized Epstein functions have trivial zeros precisely at those points in which the Gamma function is singular. As pointed out in Ref. [24], the Weldon's method subtly introduces a sort of counterterm that removes the poles from the inhomogeneous Epstein function. A remarkable advantage of this method is that it allows us to remove simultaneously both SDs and NSDs induced by the KK particles. This is a very important result because it tells us that if we regularize the Epstein functions using the Weldon's method, we should not worry about the divergences that come from KK particles, that is, our theory will be free from both NSDs and SDs associated with KK particles.

Some comments concerning the above method for the removal of divergences associated with KK particles are in order here. In principle, it is possible to keep a closer approach to the usual scheme of dealing with divergences in quantum field theory. As it has been emphasized, our SM extension to extra dimensions is not renormalizable in the Dyson's sense, but it is renormalizable in a modern sense [3–8]. So, we can remove from SGFs both NSDs and SDs coming from KK particles by introducing the appropriate counterterms (see discussion at the end of subsection II B). Let us to outline how we could proceed in this case. It must be recalled that the counterterms to remove SDs from SGFs are already present in the conventional effective Lagrangian $\mathcal{L}_{\text{eff}}^{(0)}$. Such counterterms can be designed to remove not only the SDs generated by SM fields and their KK excitations, but also to remove, in addition, the NSDs. From this perspective, to the $\sum_{(m)} \mathcal{A}_{\text{loop}}^{(m)}$ amplitude generated by the $\mathcal{L}_{\text{eff}}^{(0)(m)}$ Lagrangian we must add the amplitude $\sum_{(m)} \mathcal{A}_{\text{c.t.}}^{(m)}$ generated by the counterterm $\mathcal{L}_{\text{c.t.}}$. Besides having SDs, separately both amplitudes can have NSDs. However, when we add the counterterm amplitude to the loop amplitude, the nested infinite sums $\sum_{(m)} (\mathcal{A}_{\text{loop}}^{(m)} + \mathcal{A}_{\text{c.t.}}^{(m)})$ converge. It should be kept in mind that the loop amplitude is proportional to products of Gamma and Epstein functions, $\Gamma(N - D/2) E_l^{c^2}(N - D/2)$, and that there is a counterterm that is designed, among other things, to cancel the divergences that arise in the $D \rightarrow 4$ limit. This means that, in practice, one can implement a reduction process for the Epstein functions appearing in the loop amplitude, which consists in expressing multi-dimensional inhomogeneous Epstein functions in terms of the one-dimensional inhomogeneous function through the following relation [24]:

$$E_l^{c^2}(s) = \frac{(-1)^{l-1}}{2^{l-1}} \frac{1}{\Gamma(s)} \sum_{p=0}^{l-1} \frac{(l-1)!}{p!(l-1-p)!} \Gamma(s - \frac{p}{2}) E_1^{c^2}(s - \frac{p}{2}), \quad (\text{II.36})$$

where the one-dimensional function can be expressed in turn as follows [40]:

$$E_1^{c^2}(s) = -\frac{(c^2)^{-s}}{2} + \frac{\sqrt{\pi} \Gamma(s - \frac{1}{2})}{2 \Gamma(s)} (c^2)^{-s+\frac{1}{2}} + \frac{2\pi^s (c^2)^{-\frac{s}{2}+\frac{1}{4}}}{\Gamma(s)} \sum_{n=1}^{\infty} n^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi n \sqrt{c^2}), \quad (\text{II.37})$$

where the $K_{s-\frac{1}{2}}(2\pi n \sqrt{c^2})$ are the modified Bessel functions of second kind. The NSDs arise through the pole of $E_1^{c^2}(s)$ at $s = 1/2$, which are isolated by the simple pole of the Gamma function at $s - 1/2 = 0$. The divergences arising from this pole can be removed by the corresponding counterterm. A judicious renormalization prescription would allow us to remove, in addition, possible nondecoupling effects commonly arising in association with ultraviolet divergences (SDs). As it is discussed in Ref. [24], the Weldon's method actually cancel exactly the pole of the Epstein function, so both methods would have in common this fact. As we will see below, the Weldon's method does not remove nondecoupling effects, which always arise together with SDs, so the counterterm needed to remove the ultraviolet divergences induced by the SM fields must include a component that remove, in addition, such nonphysical effects. So, besides an irrelevant contribution proportional to the Λ scale, we can expect that both methods are equivalent. Although in principle this renormalization scheme of NSDs would work well, we prefer to use the regularized (finite) Epstein functions that emerge from the method developed by authors of Refs. [24, 25], since, as it will see below, it offers many advantages in practice.

It is important to stress that the above results depend crucially on the energy scale to which the one-loop amplitudes are analyzed. The situation may change radically at very high energies. In such a scenario the masses of the theory can be ignored, so the expression given by Eq.(II.27) becomes

$$F_N = (-1)^N \left(\frac{1}{4\pi\mu^2} \right)^{(N-2)} I_N \Gamma \left(N - \frac{D}{2} \right) \left(\frac{\hat{\Delta}^2}{4\pi\mu^2} \right) \left(\sum_{(m)} \right), \quad (\text{II.38})$$

where $\hat{\Delta}^2$ is a function that depends only on external momenta and parametric variables x_i . This limit case, which does not require of Weldon's method, is central to study the short-distance behavior of KK fields. In this case, we have the counterintuitive result

$$\begin{aligned} \sum_{(\underline{m})} &= n \sum_{m_1=1}^{\infty} + \frac{n(n-1)}{2!} \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} + \cdots + \sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} \\ &= \sum_{l=1}^n \binom{n}{l} [\zeta(0)]^l \\ &= \frac{1-2^n}{2^n}, \end{aligned} \quad (\text{II.39})$$

where the result for the Riemann zeta function $\zeta(0) = -1/2$ was used. As we will see in Sec. III, this result has interesting consequences when exploring short-distance effects through the renormalization group.

We now proceed to analyze the one-loop structure of some scalar functions given by Eq.(II.29) using the regularized Epstein functions (II.34). We focus on the following Passarino-Veltman scalar functions:

$$\begin{aligned} F_1 &= \sum_{(\underline{m})} A_0[m_{\varphi(\underline{m})}^2] \\ &= -(R^{-2}) \left(\frac{R^{-2}}{4\pi\mu^2} \right)^{(\frac{D}{2}-2)} \sum_{l=1}^n \binom{n}{l} \Gamma\left(1 - \frac{D}{2}\right) E_l^{c_1^2} \left(1 - \frac{D}{2}\right), \end{aligned} \quad (\text{II.40})$$

$$\begin{aligned} F_2 &= \sum_{(\underline{m})} B_0[p^2, m_{\varphi_0(\underline{m})}^2, m_{\varphi_1(\underline{m})}^2] \\ &= \left(\frac{R^{-2}}{4\pi\mu^2} \right)^{(\frac{D}{2}-2)} I_2 \sum_{l=1}^n \binom{n}{l} \Gamma\left(2 - \frac{D}{2}\right) E_l^{c_2^2} \left(2 - \frac{D}{2}\right), \end{aligned} \quad (\text{II.41})$$

$$\begin{aligned} F_3 &= \sum_{(\underline{m})} C_0[p_1^2, (p_1 - p_2)^2, p_2^2, m_{\varphi_0(\underline{m})}^2, m_{\varphi_1(\underline{m})}^2, m_{\varphi_2(\underline{m})}^2] \\ &= -(R^{-2})^{(\frac{D}{2}-3)} I_3 \sum_{l=1}^n \binom{n}{l} \Gamma\left(3 - \frac{D}{2}\right) E_l^{c_3^2} \left(3 - \frac{D}{2}\right), \end{aligned} \quad (\text{II.42})$$

$$\begin{aligned} F_4 &= \sum_{(\underline{m})} D_0[p_1^2, (p_1 - p_2)^2, (p_2 - p_3)^2, p_3^2, p_2^2, (p_1 - p_3)^2, m_{\varphi_0(\underline{m})}^2, m_{\varphi_1(\underline{m})}^2, m_{\varphi_2(\underline{m})}^2, m_{\varphi_3(\underline{m})}^2] \\ &= (R^{-2})^{(\frac{D}{2}-4)} I_4 \sum_{l=1}^n \binom{n}{l} \Gamma\left(4 - \frac{D}{2}\right) E_l^{c_4^2} \left(4 - \frac{D}{2}\right). \end{aligned} \quad (\text{II.43})$$

In the above expressions the $D \rightarrow 4$ limit is understood. On the other hand, in the $R^{-1} \rightarrow \infty$ limit, F_3 and F_4 vanish, but F_1 and F_2 do not. The presence of nondecoupling effects arising in association with ultraviolet divergences, as it is the case of the F_1 and F_2 functions, is in agreement with the decoupling theorem [41], since these effects are unobservable because they can be absorbed by renormalization. We will show below how to implement this when we study asymptotic freedom in Sec.III.

According to Eq.(II.29), it seems that one needs the l -dimensional regularized Epstein function, but actually only the one-dimensional Epstein function is required, since the former can be expressed in terms of the latter through the relation given by Eq.(II.36). The corresponding regularized version for the one-dimensional Epstein function is given

by [24]:

$$E_1^{c^2}(s) = \left\{ \begin{array}{l} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(k+s)}{k! \Gamma(s)} \zeta(2k+2s) (c^2)^k \\ \text{for } \frac{1}{2} - s \notin \mathbb{N}, -s \notin \mathbb{N} \\ \\ \sum_{\substack{k=0 \\ k \neq -s}}^{\infty} (-1)^k \frac{\Gamma(k+s)}{k! \Gamma(s)} \zeta(2k+2s) (c^2)^k \\ + \frac{(-1)^s (c^2)^{-s}}{2\pi} [H_{-s} - \log(4\pi^2 c^2)] \sin(\pi s) \\ \text{for } -s \in \mathbb{N}, \frac{1}{2} - s \notin \mathbb{N} \\ \\ \sum_{\substack{k=0 \\ k \neq \frac{1}{2}-s}}^{\infty} (-1)^k \frac{\Gamma(k+s)}{k! \Gamma(s)} \zeta(2k+2s) (c^2)^k \\ + \frac{(-1)^{s-\frac{1}{2}} \sqrt{\pi}}{2\Gamma(\frac{3}{2}-s)\Gamma(s)} \left[\psi\left(\frac{1}{2}\right) - \psi\left(\frac{3}{2}-s\right) + \log c^2 + 2\gamma \right] (c^2)^{\frac{1}{2}-s} \\ \text{for } \frac{1}{2} - s \in \mathbb{N}, -s \notin \mathbb{N} \end{array} \right. \quad (\text{II.44})$$

In the above expression, H_n is the harmonic number, which can be expressed in terms of the digamma function $\psi(a) \equiv \frac{d}{da} \log \Gamma(a)$ and the Euler-Macheroni constant γ , as follows: $H_n \equiv \psi(n+1) + \gamma$.

The scalar functions F_N are given by an infinite series in powers of c_N^2 ; its convergence is assured since c_N^2 is assumed to be less than unity. In our case, the c_N^2 functions are suppressed by a factor $(v/R^{-1})^2$, so in practice only a few of the first terms will be of physical relevance. However, some care should be taken with the presence of nondecoupling terms present in the F_1 and F_2 functions. Typically, we will retain the most significant terms of the series:

$$F_N = (-1)^N (R^{-2})^{(2-N)} I_N [\alpha_N^n + \beta_N^n \log(c_N^2) + \gamma_N^n c_N^2 + \delta_N^n c_N^2 \log(c_N^2) + \dots], \quad (\text{II.45})$$

where n is the number of extra dimensions. Note that for the case of the scalar functions F_3 and F_4 only the first two terms of the series will be enough, as they vanish in the $R^{-1} \rightarrow \infty$ limit. In contrast, more terms must be retained in the case of the F_1 and F_2 due to their divergent character. The corresponding results are displayed in Tables I, II, III, and IV.

The scalar functions studied above do not represent the most general situation that we can have in radiative corrections at the one-loop level. In fact, the Passarino-Veltman scalar functions can arise being multiplied or divided, or both things, by products of different KK masses. Consider an expression of the form

$$\mathcal{F}_N^M = \sum_{(\underline{m})} \left(m_{\varphi(\underline{m})}^2 \right)^M \frac{1}{i\pi^2} \int d^4 k \frac{1}{\left[k^2 - m_{\varphi_0(\underline{m})}^2 \right] \left[(k+p_1)^2 - m_{\varphi_1(\underline{m})}^2 \right] \cdots \left[(k+p_{N-1})^2 - m_{\varphi_{N-1}(\underline{m})}^2 \right]} \quad (\text{II.46})$$

where $m_{\varphi(\underline{m})}$ is some KK mass and M a positive or negative integer. The case $M = 0$ corresponds to the one already studied, that is, $\mathcal{F}_N^0 = F_N$. Note that the presence of a factor comprising the ratio of two KK masses always can be reduced to a combination of these two cases. Both positive, $M \equiv \hat{N}$, and negative, $M \equiv \bar{N}$, scenarios must be

n	α_1^n	β_1^n	γ_1^n	δ_1^n
1	$\frac{\zeta(3)}{2\pi^2}$	0	$\frac{1}{2} (\log(4\pi^2) - 1)$	$\frac{1}{2}$
2	$\frac{15\Gamma(-5/2)\zeta(4)}{16\pi^4}$	0	$\frac{3}{4} \left(\log(4\pi^2) + \frac{\Gamma(-3/2)\zeta(2)}{2\pi^2} - 1 \right)$	$\frac{3}{4}$
3	$\frac{7\zeta(3)}{8\pi^2} + \frac{3\zeta(5)}{16\pi^2} + \frac{15\Gamma(-5/2)\zeta(4)}{8\pi^4}$	0	$\frac{7}{8} \log(4\pi^2) - \frac{7}{8} - \frac{\zeta(3)}{8\pi^2} + \frac{3\Gamma(-3/2)\zeta(2)}{4\pi^2}$	$\frac{7}{8}$
4	$\frac{15\zeta(3)}{16\pi^2} + \frac{15\zeta(5)}{32\pi^4} + \frac{165\Gamma(-5/2)\zeta(4)}{64\pi^4} - \frac{105\Gamma(-7/2)\zeta(6)}{64\pi^6}$	0	$\frac{15}{16} \log(4\pi^2) - \frac{15}{16} - \frac{5\zeta(3)}{16\pi^2} + \frac{33\Gamma(-3/2)\zeta(2)}{32\pi^2} - \frac{15\Gamma(-5/2)\zeta(4)}{64\pi^4}$	$\frac{15}{16}$
5	$\frac{31\zeta(3)}{32\pi^2} + \frac{51\zeta(5)}{64\pi^4} + \frac{195\Gamma(-5/2)\zeta(4)}{64\pi^4} - \frac{315\Gamma(-7/2)\zeta(6)}{64\pi^6}$	0	$\frac{31}{32} \log(4\pi^2) - \frac{31}{32} - \frac{17\zeta(3)}{32\pi^2} + \frac{39\Gamma(-3/2)\zeta(2)}{32\pi^2} - \frac{45\Gamma(-5/2)\zeta(4)}{64\pi^4}$	$\frac{31}{32}$
6	$\frac{63\zeta(3)}{64\pi^2} + \frac{147\zeta(5)}{128\pi^4} + \frac{855\Gamma(-5/2)\zeta(4)}{256\pi^4} - \frac{1155\Gamma(-7/2)\zeta(6)}{128\pi^6} + \frac{9\Gamma(-9/2)\Gamma(8)\zeta(8)}{8192\pi^8}$	0	$\frac{63}{64} \log(4\pi^2) - \frac{63}{64} - \frac{49\zeta(3)}{64\pi^2} + \frac{171\Gamma(-3/2)\zeta(2)}{128\pi^2} - \frac{165\Gamma(-5/2)\zeta(4)}{128\pi^4} + \frac{7\Gamma(-7/2)\Gamma(6)\zeta(6)}{2048\pi^6}$	$\frac{63}{64}$
7	$\frac{127\zeta(3)}{128\pi^2} + \frac{3\zeta(5)}{2\pi^4} + \frac{105\zeta(9)}{1024\pi^8} + \frac{225\Gamma(-5/2)\zeta(4)}{64\pi^4} - \frac{105\Gamma(-7/2)\Gamma(6)\zeta(6)}{8\pi^6} + \frac{9\Gamma(-9/2)\Gamma(8)\zeta(8)}{2048\pi^8}$	0	$\frac{127}{128} \log(4\pi^2) - \frac{127}{128} - \frac{\zeta(3)}{\pi^2} - \frac{15\zeta(7)}{512\pi^6} + \frac{45\Gamma(-3/2)\zeta(2)}{32\pi^2} - \frac{15\Gamma(-5/2)\zeta(4)}{8\pi^4} + \frac{7\Gamma(-7/2)\Gamma(6)\zeta(6)}{512\pi^6}$	$\frac{127}{128}$
8	$\frac{255\zeta(3)}{256\pi^2} + \frac{117\zeta(5)}{64\pi^4} + \frac{945\zeta(9)}{2048\pi^8} + \frac{3705\Gamma(-5/2)\zeta(4)}{1024\pi^4} - \frac{17115\Gamma(-7/2)\zeta(6)}{1024\pi^6} + \frac{259\Gamma(-7/2)\Gamma(6)\zeta(6)}{8192\pi^6} - \frac{9\Gamma(-9/2)\Gamma(8)\zeta(8)}{32768\pi^8}$	0	$\frac{255}{256} \log(4\pi^2) - \frac{255}{256} - \frac{39\zeta(3)}{32\pi^2} - \frac{135\zeta(7)}{1024\pi^6} + \frac{741\Gamma(-3/2)\zeta(2)}{512\pi^2} - \frac{2445\Gamma(-5/2)\zeta(4)}{1024\pi^4} + \frac{7\Gamma(-7/2)\Gamma(6)\zeta(6)}{512\pi^6}$	$\frac{255}{256}$

TABLE I: The first coefficients of the scalar function F_1 as they appear in Eq.(II.45), given for several values of the number n of extra dimensions.

analyzed one at a time. In the positive scenario, we can write,

$$\begin{aligned}
\hat{\mathcal{F}}_N^{\hat{N}} &= (-1)^N (4\pi\mu^2)^{(2-\frac{D}{2})} (R^{-2})^{(\frac{D}{2}-N+\hat{N})} I_N \Gamma\left(N - \frac{D}{2}\right) \sum_{(\underline{m})} (\underline{m}^2 + c_0^2)^{\hat{N}} (\underline{m}^2 + c_N^2)^{-(N-\frac{D}{2})} \\
&= (-1)^N (4\pi\mu^2)^{(2-\frac{D}{2})} (R^{-2})^{(\frac{D}{2}-N+\hat{N})} I_N \sum_{K=0}^{\hat{N}} \binom{\hat{N}}{K} \left(N - \frac{D}{2} - 1\right)_{\hat{N}-K} (c_0^2 - c_N^2)^K \\
&\quad \times \sum_{l=1}^n \binom{n}{l} \Gamma\left(N - \frac{D}{2} + K - \hat{N}\right) E_l^{c_N^2} \left(N - \frac{D}{2} + K - \hat{N}\right), \tag{II.47}
\end{aligned}$$

where $c_0^2 = m_{\varphi(\mathbb{W})}^2/R^{-2}$ and $(N - \frac{D}{2} - 1)_{\hat{N}-K}$ stands for the descending factorial, which is defined by

$$(x)_n = x(x-1)(x-2)\cdots(x-n+1) = \frac{\Gamma(x+1)}{\Gamma(x-n+1)}, \tag{II.48}$$

with n a non-negative integer. In obtaining Eq.(II.47), we have used the binomial formula to express $(\underline{m}^2 + c_0^2)^{\hat{N}}$ as products of the $(\underline{m}^2 + c_N^2)$ and $(c_0^2 - c_N^2)$ factors. Due to the decoupling nature of the theory, it would be expected that terms with $\hat{N} > 1$ do not arise. In the case $\hat{N} = 1$, we have

$$\begin{aligned}
\hat{\mathcal{F}}_N^1 &= (-1)^N (R^{-2})^{(3-N)} \left(\frac{R^{-2}}{4\pi\mu^2}\right)^{(\frac{D}{2}-2)} I_N \sum_{l=1}^n \binom{n}{l} \left\{ \left(N - \frac{D}{2} - 1\right) \Gamma\left(N - \frac{D}{2} - 1\right) E_l^{c_N^2} \left(N - \frac{D}{2} - 1\right) \right. \\
&\quad \left. + (c_0^2 - c_N^2) \sum_{l=1}^n \Gamma\left(N - \frac{D}{2}\right) E_l^{c_N^2} \left(N - \frac{D}{2}\right) \right\}. \tag{II.49}
\end{aligned}$$

We now proceed to show results for the cases $N = 2, 3, 4$. Recalling that products of the way $\Gamma(s)E_l^{c^2}(s)$ are, in all

n	α_2^n	β_2^n	γ_2^n	δ_2^n
1	$-\frac{1}{2} \log(4\pi^2)$	$-\frac{1}{2}$	$-\zeta(2)$	0
2	$-\frac{3}{4} \log(4\pi^2) - \frac{3\Gamma(-\frac{3}{2})\zeta(2)}{8\pi^2}$	$-\frac{3}{4}$	$-\frac{1}{4}\Gamma(1/2) + \frac{1}{2}\gamma\Gamma(1/2)$ $-\frac{1}{4}\Gamma(1/2) \log(4) - \frac{3}{2}\zeta(2)$	$\frac{1}{4}\Gamma(\frac{1}{2})$
3	$-\frac{7}{4} \log(4\pi^2) + \frac{\zeta(3)}{8\pi^2}$ $-\frac{3\Gamma(-3/2)\zeta(2)}{4\pi^2}$	$-\frac{7}{8}$	$-\frac{1}{8} - \frac{1}{2}\Gamma(1/2)$ $-\frac{1}{2}\Gamma(1/2) \log(4)$ $+\frac{1}{8} \log(4\pi^2) - \frac{7}{4}\zeta(2)$	$\frac{1}{2}\Gamma(1/2) + \frac{1}{8}$
4	$-\frac{15}{16} \log(4\pi^2) + \frac{5\zeta(3)}{16\pi^2}$ $-\frac{33\Gamma(-3/2)\zeta(2)}{32\pi^2} + \frac{15\Gamma(-5/2)\zeta(4)}{64\pi^4}$	$-\frac{15}{16}$	$-\frac{5}{16} - \frac{11}{16}\Gamma(1/2) + \frac{11}{8}\gamma\Gamma(1/2)$ $-\frac{11}{16}\Gamma(1/2) \log(4) + \frac{5}{16} \log(4\pi^2)$ $-\frac{15}{8}\zeta(2) + \frac{3\Gamma(-3/2)\zeta(2)}{32\pi^2}$	$\frac{11}{16}\Gamma(1/2) + \frac{5}{16}$
5	$-\frac{31}{32} \log(4\pi^2) + \frac{\zeta(3)}{2\pi^2} + \frac{3\zeta(5)}{64\pi^4}$ $-\frac{39\Gamma(-3/2)\zeta(2)}{32\pi^2} + \frac{45\Gamma(-5/2)\zeta(4)}{64\pi^4}$	$-\frac{31}{32}$	$-\frac{1}{2} - \frac{13}{16}\Gamma(1/2) + \frac{13}{8}\gamma\Gamma(1/2)$ $-\frac{13}{16}\Gamma(1/2) \log(4) + \frac{1}{2} \log(4\pi^2)$ $-\frac{\zeta(3)}{32\pi^2} - \frac{31}{16}\zeta(2) + \frac{9\Gamma(-3/2)\zeta(2)}{32\pi^2}$	$\frac{13}{16}\Gamma(1/2) + \frac{1}{2}$
6	$-\frac{63}{64} \log(4\pi^2) + \frac{21\zeta(3)}{32\pi^2} + \frac{21\zeta(5)}{128\pi^4}$ $-\frac{171\Gamma(-3/2)\zeta(2)}{128\pi^2} + \frac{165\Gamma(-5/2)\zeta(4)}{128\pi^4}$ $-\frac{105\Gamma(-7/2)\zeta(6)}{256\pi^6}$	$-\frac{63}{64}$	$-\frac{21}{32} - \frac{57}{64}\Gamma(1/2) + \frac{57}{32}\gamma\Gamma(1/2)$ $-\frac{57}{64}\Gamma(1/2) \log(4) + \frac{21}{32} \log(4\pi^2)$ $-\frac{7\zeta(3)}{64\pi^2} - \frac{63}{32}\zeta(2) + \frac{33\Gamma(-3/2)\zeta(2)}{64\pi^2}$ $-\frac{15\Gamma(-5/2)\zeta(4)}{256\pi^4}$	$\frac{57}{64}\Gamma(1/2) + \frac{21}{32}$
7	$-\frac{127}{128} \log(4\pi^2) + \frac{99\zeta(3)}{128\pi^2} + \frac{87\zeta(5)}{256\pi^4}$ $+\frac{15\zeta(7)}{512\pi^6} - \frac{45\Gamma(-3/2)\zeta(2)}{32\pi^2}$ $+\frac{15\Gamma(-5/2)\zeta(4)}{8\pi^4} - \frac{105\Gamma(-7/2)\zeta(6)}{64\pi^6}$	$-\frac{127}{128}$	$-\frac{99}{128} - \frac{15}{16}\Gamma(1/2) + \frac{15}{8}\gamma\Gamma(1/2)$ $-\frac{15}{16}\Gamma(1/2) \log(4) + \frac{99}{128} \log(4\pi^2)$ $-\frac{29\zeta(3)}{128\pi^2} - \frac{3\zeta(5)}{256\pi^4} - \frac{127}{64}\zeta(2)$ $+\frac{3\Gamma(-3/2)\zeta(2)}{4\pi^2} - \frac{15\Gamma(-5/2)\zeta(4)}{64\pi^4}$	$\frac{15}{16}\Gamma(1/2) + \frac{99}{128}$
8	$-\frac{255}{256} \log(4\pi^2) + \frac{219\zeta(3)}{256\pi^2}$ $+\frac{279\zeta(5)}{512\pi^4} - \frac{741\Gamma(-3/2)\zeta(2)}{512\pi^2}$ $+\frac{135\zeta(7)}{1024\pi^6} + \frac{2445\Gamma(-5/2)\zeta(4)}{1024\pi^4}$ $-\frac{3885\Gamma(-7/2)\zeta(6)}{1024\pi^6} + \frac{9\Gamma(-9/2)\Gamma(8)\zeta(8)}{32768\pi^8}$	$-\frac{255}{256}$	$-\frac{219}{256} - \frac{247}{256}\Gamma(1/2) + \frac{247}{128}\gamma\Gamma(1/2)$ $-\frac{247}{256}\Gamma(1/2) \log(4) + \frac{219}{256} \log(4\pi^2)$ $-\frac{93\zeta(3)}{256\pi^2} - \frac{27\zeta(5)}{512\pi^4} - \frac{255}{128}\zeta(2)$ $+\frac{489\Gamma(-3/2)\zeta(2)}{512\pi^2} - \frac{555\Gamma(-5/2)\zeta(4)}{1024\pi^4}$ $+\frac{7\Gamma(-7/2)\Gamma(6)\zeta(6)}{8192\pi^6}$	$\frac{247}{256}\Gamma(1/2) + \frac{219}{256}$

TABLE II: The first coefficients of the scalar function F_2 as they appear in Eq.(II.45), given for several values of the number n of extra dimensions.

cases, finite, we can write:

$$\begin{aligned}
\hat{\mathcal{F}}_2^1 &= \sum_{(\underline{m})} m_{\varphi(\underline{m})}^2 B_0[p^2, m_{\varphi_0(\underline{m})}^2, m_{\varphi_1(\underline{m})}^2] \\
&= R^{-2} I_2 \sum_{l=1}^n \binom{n}{l} \left[-\Gamma\left(1 - \frac{D}{2}\right) E_l^{c_2^2} \left(1 - \frac{D}{2}\right) + (c_0^2 - c_2^2) \Gamma\left(2 - \frac{D}{2}\right) E_l^{c_2^2} \left(2 - \frac{D}{2}\right) \right] \\
&= R^{-2} I_2 \left\{ (c_0^2 - c_2^2) \alpha_2^n - \alpha_1^n + (c_0^2 \beta_2^n - \beta_1^n) \log(c_2^2) + [(c_0^2 - c_2^2) \gamma_2^n - \gamma_1^n] c_2^2 + \dots \right\}, \tag{II.50}
\end{aligned}$$

where we have made use of the fact that $\delta_1^n = -\beta_2^n$. Note that $\hat{\mathcal{F}}_2^1$ diverges in the $R^{-2} \rightarrow \infty$ limit. On the other

n	α_3^n	β_3^n
1	$-2\pi^2\zeta(-1)$	0
2	$\frac{1}{4}\Gamma(1/2)\log(4) - \frac{1}{2}\gamma\Gamma(1/2) - 3\pi^2\zeta(-1)$	$-\frac{1}{4}\Gamma(1/2)$
3	$\frac{1}{2}\Gamma(1/2)\log(4) - \gamma\Gamma(1/2) - \frac{7}{2}\pi^2\zeta(-1)$ $-\frac{1}{8}\log(4\pi^2)$	$-\frac{1}{2}\Gamma(1/2) - \frac{1}{8}$
4	$\frac{11}{16}\Gamma(1/2)\log(4) - \frac{11}{8}\gamma\Gamma(1/2) - \frac{15}{4}\pi^2\zeta(-1)$ $-\frac{5}{16}\log(4\pi^2) - \frac{3\Gamma(-3/2)\zeta(2)}{32\pi^2}$	$-\frac{11}{16}\Gamma(1/2) - \frac{5}{16}$
5	$\frac{13}{16}\Gamma(1/2)\log(4) - \frac{13}{8}\gamma\Gamma(1/2) - \frac{31}{8}\pi^2\zeta(-1)$ $-\frac{1}{2}\log(4\pi^2) - \frac{9\Gamma(-3/2)\zeta(2)}{32\pi^2} + \frac{\zeta(3)}{32\pi^2}$	$-\frac{13}{16}\Gamma(1/2) - \frac{1}{2}$
6	$\frac{57}{64}\Gamma(1/2)\log(4) - \frac{57}{32}\gamma\Gamma(1/2) - \frac{63}{16}\pi^2\zeta(-1)$ $-\frac{21}{32}\log(4\pi^2) - \frac{33\Gamma(-3/2)\zeta(2)}{64\pi^2} + \frac{7\zeta(3)}{64\pi^2}$ $+\frac{15\Gamma(-5/2)\zeta(4)}{256\pi^4}$	$-\frac{57}{64}\Gamma(1/2) - \frac{21}{32}$
7	$\frac{15}{16}\Gamma(1/2)\log(4) - \frac{15}{8}\gamma\Gamma(1/2) - \frac{127}{32}\pi^2\zeta(-1)$ $-\frac{99}{128}\log(4\pi^2) - \frac{3\Gamma(-3/2)\zeta(2)}{4\pi^2} + \frac{29\zeta(3)}{128\pi^2}$ $+\frac{15\Gamma(-5/2)\zeta(4)}{64\pi^4} + \frac{3\zeta(5)}{256\pi^4}$	$-\frac{15}{16}\Gamma(1/2) - \frac{99}{128}$
8	$\frac{247}{256}\Gamma(1/2)\log(4) - \frac{247}{128}\gamma\Gamma(1/2) - \frac{255}{64}\pi^2\zeta(-1)$ $-\frac{219}{256}\log(4\pi^2) - \frac{489\Gamma(-3/2)\zeta(2)}{512\pi^2} + \frac{93\zeta(3)}{256\pi^2}$ $+\frac{555\Gamma(-5/2)\zeta(4)}{1024\pi^4} + \frac{27\zeta(5)}{512\pi^4} - \frac{7\Gamma(-7/2)\Gamma(6)\zeta(6)}{8192\pi^6}$	$-\frac{247}{256}\Gamma(1/2) - \frac{219}{256}$

TABLE III: The first coefficients of the scalar function F_3 as they appear in Eq.(II.45), given for several values of the number n of extra dimensions.

n	α_4^n	β_4^n
1	$\frac{4}{3}\pi^4\zeta(-3)$	0
2	$2\pi^4\zeta(-3) - \frac{1}{4}\Gamma(1/2)\zeta(3)$	0
3	$-\frac{1}{2}\Gamma(1/2)\zeta(3) + \frac{7}{3}\pi^4\zeta(-3) - \frac{1}{2}\pi^2\zeta(-1)$	0
4	$-\frac{1}{8}\gamma\Gamma(1/2) + \frac{1}{16}\Gamma(1/2) - \frac{11}{16}\Gamma(1/2)\zeta(3)$ $+\frac{5}{2}\pi^4\zeta(-3) - \frac{5}{4}\pi^2\zeta(-1)$	$-\frac{1}{16}\Gamma(1/2)\log(4)$
5	$-\frac{3}{8}\gamma\Gamma(1/2) - \frac{3}{16}\Gamma(1/2)\log(4) - \frac{1}{32}\log(4\pi^2)$ $-\frac{13}{16}\Gamma(1/2)\zeta(3) + \frac{31}{12}\pi^4\zeta(-3) - 2\pi^2\zeta(-1)$	$-\frac{3}{16}\Gamma(1/2) - \frac{1}{32}$
6	$-\frac{11}{16}\gamma\Gamma(1/2) + \frac{11}{32}\Gamma(1/2)\log(4) - \frac{7}{64}\log(4\pi^2)$ $-\frac{57}{64}\Gamma(1/2)\zeta(3) + \frac{21}{8}\pi^4\zeta(-3) - \frac{21}{8}\pi^2\zeta(-1)$ $-\frac{3\Gamma(-3/2)\zeta(2)}{128\pi^2}$	$-\frac{57}{64}\Gamma(1/2) - \frac{21}{32}$
7	$-\gamma\Gamma(1/2) + \frac{1}{2}\Gamma(1/2)\log(4) - \frac{29}{128}\log(4\pi^2)$ $-\frac{15}{16}\Gamma(1/2)\zeta(3) + \frac{127}{48}\pi^4\zeta(-3) - \frac{99}{32}\pi^2\zeta(-1)$ $-\frac{3\Gamma(-3/2)\zeta(2)}{32\pi^2} + \frac{\zeta(3)}{128\pi^2}$	$-\frac{1}{2}\Gamma(1/2) - \frac{29}{128}$
8	$-\frac{163}{128}\gamma\Gamma(1/2) + \frac{163}{256}\Gamma(1/2)\log(4) - \frac{93}{256}\log(4\pi^2)$ $-\frac{247}{256}\Gamma(1/2)\zeta(3) + \frac{85}{32}\pi^4\zeta(-3) - \frac{219}{64}\pi^2\zeta(-1)$ $-\frac{111\Gamma(-3/2)\zeta(2)}{512\pi^2} + \frac{9\zeta(3)}{256\pi^2} + \frac{15\Gamma(-5/2)\zeta(4)}{1024\pi^4}$	$-\frac{163}{256}\Gamma(1/2) - \frac{93}{256}$

TABLE IV: The first coefficients of the scalar function F_4 as they appear in Eq.(II.45), given for several values of the number n of extra dimensions.

hand, for $N = 3$ one has

$$\begin{aligned}
\hat{\mathcal{F}}_3^1 &= \sum_{(\underline{m})} m_{\varphi(\underline{m})}^2 C_0[p_1^2, (p_1 - p_2)^2, p_2^2, m_{\varphi_0(\underline{m})}^2, m_{\varphi_1(\underline{m})}^2, m_{\varphi_2(\underline{m})}^2] \\
&= -I_3 \sum_{l=1}^n \binom{n}{l} \left\{ \left(2 - \frac{D}{2}\right) \Gamma\left(2 - \frac{D}{2}\right) E_l^{c_3^2} \left(2 - \frac{D}{2}\right) + (c_0^2 - c_3^2) \Gamma\left(3 - \frac{D}{2}\right) E_l^{c_3^2} \left(3 - \frac{D}{2}\right) \right\} \\
&= -I_3 \left\{ 2 \left(\frac{2^n - 1}{2^n} \right) \zeta(2) c_3^2 + (c_0^2 - c_3^2) [\alpha_3^n + \beta_3^n \log(c_3^2) + \dots] \right\}.
\end{aligned} \tag{II.51}$$

Note that in this case $\hat{\mathcal{F}}_3^1$ disappears in the $R^{-2} \rightarrow \infty$ limit, so it is not necessary to keep terms that involve higher powers of c_3^2 . Finally, the expression corresponding to $N = 4$ is given by

$$\begin{aligned}
\hat{\mathcal{F}}_4^1 &= \sum_{(\underline{m})} m_{\varphi(\underline{m})}^2 D_0[p_1^2, (p_1 - p_2)^2, (p_2 - p_3)^2, p_3^2, p_2^2, (p_1 - p_3)^2, m_{\varphi_0(\underline{m})}^2, m_{\varphi_1(\underline{m})}^2, m_{\varphi_2(\underline{m})}^2, m_{\varphi_3(\underline{m})}^2] \\
&= \frac{1}{R^{-2}} I_4 \sum_{l=1}^n \binom{n}{l} \left[\Gamma\left(3 - \frac{D}{2}\right) E_l^{c_4^2} \left(3 - \frac{D}{2}\right) + (c_0^2 - c_4^2) \Gamma\left(4 - \frac{D}{2}\right) E_l^{c_4^2} \left(4 - \frac{D}{2}\right) \right] \\
&= \frac{1}{R^{-2}} I_4 \{ \alpha_3^n + (c_0^2 - c_4^2) \alpha_4^n + [\beta_3^n + (c_0^2 - c_4^2) \beta_4^n] \log(c_4^2) + \dots \}.
\end{aligned} \tag{II.52}$$

We now turn to analyze the negative scenario. We will focus on those amplitudes which are free of ultraviolet divergences, since they would be found more frequently. Therefore, in this part of our analysis, we will assume $D = 4$. In this case, we use a sort of Feynman parametrization to write

$$\begin{aligned}
\frac{1}{(\underline{m}^2 + c_0^2)^{\bar{N}} (\underline{m}^2 + c_N^2)^{(N-2)}} &= \frac{\Gamma(N-2+\bar{N})}{\Gamma(\bar{N}) \Gamma(N-2)} \int_0^1 dz \frac{z^{(N-3)} (1-z)^{(\bar{N}-1)}}{[(\underline{m}^2 + c_N^2)z + (\underline{m}^2 + c_0^2)(1-z)]^{(N-2+\bar{N})}} \\
&= \frac{\Gamma(N-2+\bar{N})}{\Gamma(\bar{N}) \Gamma(N-2)} \int_0^1 dz \frac{z^{(N-3)} (1-z)^{(\bar{N}-1)}}{(\underline{m}^2 + c_{N0}^2)^{(N-2+\bar{N})}},
\end{aligned} \tag{II.53}$$

where

$$c_{N0}^2 = c_N^2 z + c_0^2 (1-z). \tag{II.54}$$

Then, we can write for the negative case

$$\bar{\mathcal{F}}_N^{\bar{N}} = (-1)^N (R^{-2})^{(2-N-\bar{N})} \bar{I}_{\bar{N}} I_N \sum_{l=1}^n \binom{n}{l} \Gamma(N-2+\bar{N}) E_l^{c_{N0}^2} (N-2+\bar{N}), \tag{II.55}$$

where

$$\bar{I}_{\bar{N}} = \frac{1}{\Gamma(\bar{N})} \int_0^1 dz z^{(N-3)} (1-z)^{(\bar{N}-1)}. \tag{II.56}$$

We shall analyze the cases $\bar{N} = 1, 2$ for $N = 3, 4$. Before analyzing the cases of C_0 and D_0 functions divided by powers of KK masses, let us pause for a moment to discuss the especial case of B_0 functions appearing in the way

$$\Delta B_0 = B_0[p_1^2, m_{\varphi_1(\underline{m})}^2, m_{\varphi_2(\underline{m})}^2] - B_0[p_2^2, m_{\varphi_3(\underline{m})}^2, m_{\varphi_4(\underline{m})}^2], \tag{II.57}$$

which is common in radiative corrections. This expression, which can be arranged in a linear combination of two terms with 3 propagators, can be written as follows:

$$\mathcal{F}_{3, \Delta B_0}^{\bar{N}} = -\frac{1}{(R^{-2})^{(1+\bar{N})}} I_3 \bar{I}_{\bar{N}} \sum_{l=1}^n \binom{n}{l} \left[f(x_1, x_2) E_l^{a_{30}^2} (1+\bar{N}) + (m_{\varphi_1(\underline{m})}^2 - m_{\varphi_3(\underline{m})}^2) E_l^{b_{30}^2} (1+\bar{N}) \right], \tag{II.58}$$

where

$$a_{30}^2 = a_3^2 z + c_0^2 (1-z), \tag{II.59}$$

$$b_{30}^2 = b_3^2 z + c_0^2 (1-z), \tag{II.60}$$

with

$$a_3^2 = \frac{1}{R^{-2}} \left[m_{\phi_3^{(\omega)}}^2 + \left(m_{\phi_2^{(\omega)}}^2 - m_{\phi_3^{(\omega)}}^2 \right) x_1 + \left(m_{\phi_4^{(\omega)}}^2 - m_{\phi_3^{(\omega)}}^2 \right) x_2 - p_1^2 x_1 (1 - x_1) - p_2^2 x_2 (1 - x_2) + 2p_1 \cdot p_2 x_1 x_2 \right], \quad (\text{II.61})$$

$$b_3^2 = \frac{1}{R^{-2}} \left[m_{\phi_1^{(\omega)}}^2 + \left(m_{\phi_2^{(\omega)}}^2 - m_{\phi_1^{(\omega)}}^2 \right) x_1 + \left(m_{\phi_3^{(\omega)}}^2 - m_{\phi_1^{(\omega)}}^2 \right) x_2 - p_1^2 x_1 (1 - x_1) \right]. \quad (\text{II.62})$$

In addition,

$$f(x_1, x_2) = 2 \left[(p_1^2 - p_1 \cdot p_2) x_1 - (p_2^2 - p_1 \cdot p_2) x_2 \right] + p_2^2 - p_1^2 + m_{\phi_2^{(\omega)}}^2 - m_{\phi_4^{(\omega)}}^2. \quad (\text{II.63})$$

As it can be appreciated from the global coefficient in (II.58), these effects are quite suppressed because they are decreased as R^4 and R^6 for $\bar{N} = 1, 2$, respectively. So, we will keep only the first term in the power series for the Epstein function. In this limit, the integrals $\bar{I}_{\bar{N}}$ and I_3 have simple solutions. Therefore, Eq.(II.58) becomes:

$$\mathcal{F}_{3, \Delta B_0}^{\bar{N}} = -\frac{1}{(R^{-2})^{(1+\bar{N})}} \frac{1}{\Gamma(\bar{N})} \left[\frac{1}{6} (p_2^2 - p_1^2) + \frac{1}{2} \left(m_{\phi_1^{(\omega)}}^2 + m_{\phi_2^{(\omega)}}^2 - m_{\phi_3^{(\omega)}}^2 - m_{\phi_4^{(\omega)}}^2 \right) \right] \bar{\alpha}_{\bar{N}}^n, \quad (\text{II.64})$$

where

$$\bar{\alpha}_{\bar{N}}^n = \sum_{l=1}^n \binom{n}{l} E_l^{(a_{30}^2, b_{30}^2) \rightarrow 0} (1 + \bar{N}). \quad (\text{II.65})$$

In Tables V and VI, we list the corresponding values of $\bar{\alpha}_1^n$ and $\bar{\alpha}_2^n$ for several values of the number of extra dimensions n .

TABLE V: $\bar{\alpha}_1^n$ as a function of the number n of extra dimensions.

n	$\bar{\alpha}_1^n$
1	$\zeta(4)$
2	$\frac{3}{2} \zeta(4) - \frac{1}{4} \Gamma(\frac{1}{2}) \zeta(3)$
3	$\frac{1}{4} \zeta(2) + \frac{1}{4} \zeta(4) - \frac{1}{2} \Gamma(\frac{1}{2}) \zeta(3)$
4	$-\frac{1}{16} (\gamma + \psi(\frac{1}{2})) \Gamma(\frac{1}{2}) + \frac{5}{8} \zeta(2) + \frac{15}{8} \zeta(4) - \frac{11}{16} \Gamma(\frac{1}{2}) \zeta(3)$
5	$\frac{1}{16} \left[\left(\frac{1}{6} - 9\gamma - 3\psi(\frac{1}{2}) \right) \Gamma(\frac{1}{2}) + 16\zeta(2) + 31\zeta(4) - 13\Gamma(\frac{1}{2}) \zeta(3) \right]$
6	$\frac{1}{64} \left[\left(\frac{7}{3} - 198\gamma - 66\psi(\frac{1}{2}) \right) \Gamma(\frac{1}{2}) + 84\zeta(2) + 63\zeta(4) - \frac{1}{2} \frac{\zeta(3)}{\zeta(2)} \zeta(2) - 57\Gamma(\frac{1}{2}) \zeta(3) \right]$
7	$\frac{1}{16} \left[\left(\frac{109}{90} - 24\gamma - 8\psi(\frac{1}{2}) \right) \Gamma(\frac{1}{2}) + \frac{99}{4} \zeta(2) + \frac{127}{4} \zeta(4) - \frac{1}{6} \frac{\zeta(3)}{\zeta(2)} - 15\Gamma(\frac{1}{2}) \zeta(3) \right]$
8	$\frac{1}{256} \left[\left(\frac{78}{5} - 489\gamma - 163\psi(\frac{1}{2}) \right) \Gamma(\frac{1}{2}) + 438\zeta(2) + 510\zeta(4) - \frac{370}{6} \frac{\zeta(3)}{\zeta(2)} - 247\Gamma(\frac{1}{2}) \zeta(3) - \frac{1}{60} \frac{\zeta(5)}{\zeta(4)} \right]$

TABLE VI: $\bar{\alpha}_2^n$ as a function of the number n of extra dimensions.

n	$\bar{\alpha}_2^n$
1	$\Gamma(3) \zeta(6)$
2	$3\zeta(6) - \frac{3}{8} \Gamma(\frac{1}{2}) \zeta(5)$
3	$\frac{1}{4} \zeta(4) + \frac{1}{2} \zeta(6) - \frac{3}{4} \Gamma(\frac{1}{2}) \zeta(5)$
4	$\frac{1}{16} \left[10\zeta(4) + \frac{3780}{63} \zeta(6) - \Gamma(\frac{1}{2}) \zeta(3) - \frac{33}{2} \Gamma(\frac{1}{2}) \zeta(5) \right]$
5	$\frac{1}{16} \left[\zeta(2) + 16\zeta(4) + \frac{29295}{460} \zeta(6) - 3\Gamma(\frac{1}{2}) \zeta(3) - \frac{39}{2} \Gamma(\frac{1}{2}) \zeta(5) \right]$
6	$\frac{1}{64} \left[- (3\gamma + \psi(\frac{1}{2})) \Gamma(\frac{1}{2}) + \frac{42}{3} \zeta(2) + 84\zeta(4) + 252\zeta(6) - 22\Gamma(\frac{1}{2}) \zeta(3) - \frac{171}{2} \Gamma(\frac{1}{2}) \zeta(5) \right]$
7	$\frac{1}{32} \left[-2 (3\gamma + \psi(\frac{1}{2})) \Gamma(\frac{1}{2}) - \frac{29}{2} \zeta(2) + 99\zeta(4) + 127\zeta(6) - 16\Gamma(\frac{1}{2}) \zeta(3) - 45\Gamma(\frac{1}{2}) \zeta(5) \right]$
8	$\frac{1}{256} \left[- (1 + 108\gamma + 36\psi(\frac{1}{2})) \Gamma(\frac{1}{2}) + 186\zeta(2) + 438\zeta(4) + 1020\zeta(6) - 163\Gamma(\frac{1}{2}) \zeta(3) - \frac{741}{2} \Gamma(\frac{1}{2}) \zeta(5) \right]$

We now turn to analyze the situation in which C_0 and D_0 functions appear divided by KK masses. In these cases, we can write

$$\begin{aligned}
\bar{\mathcal{F}}_3^{\bar{N}} &= \sum_{(\underline{m})} \frac{1}{\left(m_{\varphi(\underline{m})}^2\right)^{\bar{N}}} C_0[p_1^2, (p_1 - p_2)^2, p_2^2, m_{\varphi_0(\underline{m})}^2, m_{\varphi_1(\underline{m})}^2, m_{\varphi_2(\underline{m})}^2] \\
&= -\frac{1}{(R^{-2})^{(\bar{N}+1)}} \bar{I}_{\bar{N}} I_3 \sum_{l=1}^n \binom{n}{l} \Gamma(\bar{N}+1) E_l^{c_{30}^2}(\bar{N}+1),
\end{aligned} \tag{II.66}$$

$$\begin{aligned}
\bar{\mathcal{F}}_4^{\bar{N}} &= \sum_{(\underline{m})} \frac{1}{\left(m_{\varphi(\underline{m})}^2\right)^{\bar{N}}} D_0[p_1^2, (p_1 - p_2)^2, (p_2 - p_3)^2, p_3^2, p_2^2, (p_1 - p_3)^2, m_{\varphi_0(\underline{m})}^2, m_{\varphi_1(\underline{m})}^2, m_{\varphi_2(\underline{m})}^2, m_{\varphi_3(\underline{m})}^2] \\
&= \frac{1}{(R^{-2})^{(\bar{N}+2)}} \bar{I}_{\bar{N}} I_4 \sum_{l=1}^n \binom{n}{l} \Gamma(\bar{N}+2) E_l^{c_{40}^2}(\bar{N}+2).
\end{aligned} \tag{II.67}$$

Ignoring terms proportional to the c_{30}^2 and c_{40}^2 quantities in the corresponding expansions for the Epstein functions, we can write

$$\bar{\mathcal{F}}_3^{\bar{N}} = -\frac{1}{(R^{-2})^{(\bar{N}+1)}} \left(\frac{\bar{N}}{2}\right) \bar{\alpha}_{\bar{N}}^n, \tag{II.68}$$

$$\bar{\mathcal{F}}_4^{\bar{N}} = \frac{1}{(R^{-2})^{(\bar{N}+2)}} \frac{\bar{N}(\bar{N}+1)}{\bar{N}} \bar{\beta}_{\bar{N}}^n, \tag{II.69}$$

where

$$\bar{\beta}_{\bar{N}}^n = \sum_{l=1}^n \binom{n}{l} E_l^{a_{40}^2 \rightarrow 0} (2 + \bar{N}). \tag{II.70}$$

Note that $\bar{\beta}_1^n = \bar{\alpha}_2^n$. The values of $\bar{\beta}_2^n$ are displayed in Table VII.

TABLE VII: $\bar{\beta}_2^n$ as a function of the number n of extra dimensions.

n	β_2^n
1	$\Gamma(4) \zeta(8)$
2	$9 \zeta(8) - \frac{15}{16} \Gamma(\frac{1}{2}) \zeta(7)$
3	$\frac{1}{2} \zeta(6) + \frac{21}{2} \zeta(8) - \frac{15}{8} \Gamma(\frac{1}{2}) \zeta(7)$
4	$\frac{35}{28} \zeta(6) + \frac{45}{4} \zeta(8) - \frac{3}{32} \Gamma(\frac{1}{2}) \zeta(5) - \frac{165}{64} \Gamma(\frac{1}{2}) \zeta(7)$
5	$\frac{63}{32} \zeta(4) + 2 \zeta(6) + \frac{93}{8} \zeta(8) - \frac{9}{32} \Gamma(\frac{1}{2}) \zeta(5) - \frac{195}{64} \Gamma(\frac{1}{2}) \zeta(7)$
6	$\frac{7}{32} \zeta(4) + \frac{63}{24} \zeta(6) + \frac{189}{16} \zeta(8) - \frac{1}{64} \Gamma(\frac{1}{2}) \zeta(3) - \frac{33}{64} \Gamma(\frac{1}{2}) \zeta(5) - \frac{855}{256} \Gamma(\frac{1}{2}) \zeta(7)$
7	$\frac{1}{64} \zeta(2) + \frac{29}{64} \zeta(4) + \frac{693}{224} \zeta(6) + \frac{1905}{160} \zeta(8) - \frac{1}{16} \Gamma(\frac{1}{2}) \zeta(3) - \frac{3}{4} \Gamma(\frac{1}{2}) \zeta(5) - \frac{225}{64} \zeta(7)$
8	$-\frac{1}{256} (3\gamma + \psi(\frac{1}{2})) \Gamma(\frac{1}{2}) + \frac{9}{128} \zeta(2) + \frac{93}{128} \zeta(4) + \frac{1533}{448} \zeta(6) + \frac{5355}{448} \zeta(8) - \frac{37}{256} \Gamma(\frac{1}{2}) \zeta(3) - \frac{489}{512} \Gamma(\frac{1}{2}) \zeta(5) - \frac{3705}{1024} \Gamma(\frac{1}{2}) \zeta(7)$

D. Remarks on one-loop renormalization of SGFs

Let us look more closely the total contribution, up to the one-loop level, from the EDSM to a SGF, which is represented by Eq.(II.21). The tree-level contribution, given by Eq.(II.22a), is exclusively determined by the conventional effective theory $\mathcal{L}_{\text{eff}}^{(0)}$, since there are no contributions from KK particles at this level. Such a contribution can arise from both the SM, $\mathcal{L}_{\text{SM}}^{(0)}$, and effective interactions of dimension higher than four, $\mathcal{L}_{\text{d}>4}^{(0)}$. Of course, this type of contributions are free of both SDs and NSDs, but they already suggest the appropriate counterterms to absorb possible divergences and thus to implement a renormalization prescription in a modern sense, if necessary. Divergences can arise at the one-loop level from two sources, namely, the conventional effective theory, whose contribution is given by Eq.(II.22b), and the nonconventional part of the effective theory, characterized by the effects of the KK particles, whose contribution is represented by Eq.(II.22c). The infinite number of KK particles lead to both SDs and NSDs, but we have seen that they can simultaneously removed by regularizing the Epstein function using the Weldon's method. This means that, at the one-loop level, the only divergences of an arbitrary SGF are SDs induced by the

conventional effective theory, that is, ultraviolet divergences induced by couplings only among SM fields, which can be renormalizable in the Dyson's sense, $\mathcal{L}_{\text{SM}}^{(0)}$, or nonrenormalizable in this sense, $\mathcal{L}_{\text{d}>4}^{(0)}$. As we have been pointing out throughout the paper, our SM extension to extra dimensions is not renormalizable in the Dyson's sense, but it is renormalizable in a modern sense [3–8], so we can renormalize any SGF by introducing appropriate counterterms (see discussion at the end of subsection II B).

From the above discussion, it follows that the EDSM is renormalizable in a modern sense, just as it is a conventional effective theory $\mathcal{L}_{\text{eff}}^{(0)}$. However, it is important to note that the Weldon's method, although suitable for removing both SDs and NSDs, fails in removing nondecoupling effects induced by the KK particles. The presence of these nondecoupling effects can be identified in the F_1 and F_2 functions, which do not disappear in the $R^{-1} \rightarrow \infty$ limit. Of course, this type of effects cannot have a physical origin, so they must be removed through a renormalization prescription. Let us analyze in more depth the structure of this type of contribution. To this end, consider the most general case that we could find in some one-loop contribution of KK particles to a SGF. Assuming, for the sake of simplicity, that there are only logarithmic SDs, we can write

$$\Gamma_N^{\text{loop}} \left(\mathcal{L}_{\text{eff}}^{(0)(\underline{m})} \right) = \frac{i}{(4\pi)^2} f \left(\frac{R^{-2}}{4\pi\mu^2} \right)^{-(2-\frac{D}{2})} I_2 \sum_{l=1}^n \frac{n!}{l!(n-l)!} E_l^{c^2} \left(2 - \frac{D}{2} \right) \Gamma \left(2 - \frac{D}{2} \right) + \sum_{(\underline{m})} \mathcal{A}^{(\underline{m})}, \quad (\text{II.71})$$

where f is some function that does not depend on KK masses. In this expression, the $\mathcal{A}^{(\underline{m})}$ amplitude comprises contributions that are free of SDs. According to our analysis of Sec.III of Part I, the heavy KK-mass behavior of the $\mathcal{A}^{(\underline{m})}$ amplitudes can be analyzed from two perspectives, namely, by making the compactification scale R^{-1} large or by making the dimensionless parameters given by the Fourier indices large while maintaining R^{-1} fixed. The former scenario corresponds to making all the KK spectrum very heavy and eventually decoupling its effects from the SM observables; whereas the latter one has to do with making some KK particles heavier than others. Here, we focus on the former scenario. In this case, the decoupling theorem [41] tells us that, due to the presence of SDs, the KK effects on the $\mathcal{A}^{(\underline{m})}$ amplitudes do not necessarily decouple, that is,

$$\lim_{R^{-1} \rightarrow \infty} \mathcal{A}^{(\underline{m})} = \text{constant} \neq 0. \quad (\text{II.72})$$

Nondecoupling effects of this type would arise from a term of the way

$$\mathcal{A}^{(\underline{m})} \sim \sum_{(\underline{m})} m_{\varphi_0}^2 \left(B_0[p_1^2, m_{\varphi_1}^2, m_{\varphi_2}^2] - B_0[p_2^2, m_{\varphi_3}^2, m_{\varphi_4}^2] \right), \quad (\text{II.73})$$

which are typical of Green's functions containing ultraviolet divergences. It is well known that the presence of nondecoupling effects that arise together with ultraviolet divergences are compatible with the decoupling theorem [41], as they can be absorbed by renormalization. This is quite common in deriving conventional effective theories by integrating out heavy fields (see, for instance, Ref. [42]). In our case, it is not only needed to remove this type of nondecoupling effects, but also those arising after implementing the Weldon's method, which, in the case of logarithmic divergences, are encoded in the F_2 function. However, any SFG that have SDs induced by KK particles, also have SDs induced by the SM particles, which must be removed by renormalization. The chosen renormalization scheme must be able to cancel the SDs generated by the SM fields at the same time that it removes the nondecoupling effects. Below, we will discuss how to do this in the context of Yang-Mills theories at the one-loop level. In other words, the nondecoupling effects must be removed by the same counterterm needed to renormalize, in a modern sense (Dyson's sense), the contribution generated by the conventional effective theory $\mathcal{L}_{\text{eff}}^{(0)}$ (SM $\mathcal{L}_{\text{SM}}^{(0)}$).

In the EDSM, some processes can be quite sensitive to KK particles, as they first occur at the one-loop level within the SM. To be specific, imagine a SM process which is free of SDs because it first occurs at the one-loop level as, for instance, flavor changing neutral currents decays of the top quark or the Higgs boson, or the rare decays of the Higgs boson into two photons or a photon and a Z boson. The one-loop contribution of KK particles to any of these processes would be characterized by an amplitude of the way

$$\mathcal{A} = \mathcal{A}^{(0)} + \sum_{(\underline{m})} \mathcal{A}^{(\underline{m})}, \quad (\text{II.74})$$

where $\mathcal{A}^{(0)}$ is the SM contribution, which is free of SDs. Since there are no SDs, the decoupling theorem says that

$$\lim_{R^{-1} \rightarrow \infty} \mathcal{A}^{(\underline{m})} = 0. \quad (\text{II.75})$$

That is, the entire KK spectrum decouples in this limit.

From the above discussion, it is clear that physical processes as the one represented by the (II.74) amplitude are free not only from divergences, but also from nondecoupling effects. So, renormalization is not necessary for this type of processes if regularized Epstein functions are used.

In the next section, we will follow the above ideas in order to implement a renormalization prescription for Yang-Mills theories at the one-loop level.

III. ASYMPTOTIC FREEDOM

It is a well-known fact that non-Abelian gauge theories present the phenomenon of asymptotic freedom [44]. Asymptotic freedom means that at shorter distances, the coupling constant decreases in size, so that the theory appears to be a free theory. This dynamical principle, which is central for understanding the strong interaction, is studied through renormalization-group techniques. Asymptotic freedom means that the β function of a non-Abelian gauge theory with a sufficiently small number of fermions is negative. The purpose of this section is to investigate whether this result remains true if extra dimensions exist.

A. The effective action

Our main goal is to calculate, at the one-loop order, the β function of the gauge group $SU(N, \mathcal{M}^4)$. To perform this calculation, we are interested in those couplings that contribute at one-loop to SGFs in which the external fields are the zero mode gauge fields $A_\mu^{(0)a}$. Such terms include KK excited modes of extra-dimensional gauge fields, $A_\mu^{(\underline{m})a}$ and its associated scalar fields $A_n^{(\underline{m})a}$, and fermions fields $F^{(\underline{m})}$ (see Sec.III, Sec.IVA1, and Sec.IVB of Part I), but note that their zero modes must be considered as well. So, our task is to compute the effective action $\Gamma[A^{(0)}]$ resulting from integrating out the quantum fluctuations of $A_\mu^{(0)a}$ and $F^{(0)}$, including their KK excitations. Gauge-fixing procedures for both the $A_\mu^{(0)a}$ and the $A_\mu^{(\underline{m})a}$ gauge fields that lead to a $\Gamma[A^{(0)}]$ action invariant under standard gauge transformations (SGTs) would be desirable because then we could compute more easily the β function. To see this, we let the renormalized field and coupling be

$$A_{B\mu}^{(0)a} = \sqrt{Z_A} A_\mu^{(0)a}, \quad g_B = \sqrt{Z_g} g, \quad (\text{III.1})$$

where $\{A_{B\mu}^{(0)a}, g_B\}$ and $\{A_\mu^{(0)a}, g\}$ stand for bare and renormalized quantities, respectively. The relation between the bare and the renormalized Yang-Mills Lagrangian is given by

$$\begin{aligned} -\frac{1}{4} F_{B\mu\nu}^{(0)a} F_{B\mu\nu}^{(0)a} &= -\frac{1}{4} F_{\mu\nu}^{(0)a} F_{\mu\nu}^{(0)a} - \frac{1}{4} (Z_A - 1) \left(\partial_\mu A_\nu^{(0)a} - \partial_\nu A_\mu^{(0)a} \right) \left(\partial^\mu A^{(0)a\nu} - \partial^\nu A^{(0)a\mu} \right) \\ &\quad - \frac{1}{2} g \left(\sqrt{Z_g} Z_A^{3/2} - 1 \right) \left(\partial_\mu A_\nu^{(0)a} - \partial_\nu A_\mu^{(0)a} \right) f^{abc} A^{(0)b\mu} A^{(0)c\nu} \\ &\quad - \frac{1}{4} g^2 \left(Z_g Z_A^2 - 1 \right) f^{abc} A_\mu^{(0)b} A_\nu^{(0)c} f^{ade} A^{(0)d\mu} A^{(0)e\nu}. \end{aligned} \quad (\text{III.2})$$

To determine Z_A , at a given order, it is enough to compute the vacuum-polarization. However, to determine Z_g and hence the renormalization of the coupling constant one needs to calculate, in addition, the three- and four-gauge boson vertex functions or, equivalently, the gauge boson-fermion-fermion function. However, if we can keep standard gauge invariance on the effective action, the relation $Z_A Z_g = 1$ would be true, so that expression (III.2) becomes

$$-\frac{1}{4} F_{B\mu\nu}^{(0)a} F_{B\mu\nu}^{(0)a} = -\frac{1}{4} F_{\mu\nu}^{(0)a} F_{\mu\nu}^{(0)a} - \frac{1}{4} \delta_A F_{\mu\nu}^{(0)a} F_{\mu\nu}^{(0)a}, \quad (\text{III.3})$$

where $\delta_A = Z_A - 1$ is the counterterm. In this case, $g_B = Z_A^{-1/2} g$. Below, we will discuss special types of gauge-fixing procedures for the $A_\mu^{(0)a}$ and the $A_\mu^{(\underline{m})a}$ gauge fields in which gauge invariance is maintained at the level of the effective action $\Gamma[A^{(0)}]$ and, hence, it is really simple to obtain the coupling constant renormalization and the β function.

We now proceed to introduce $SU(N, \mathcal{M}^4)$ -covariant gauge-fixing procedures for the $A_\mu^{(0)a}$ and $A_\mu^{(\underline{m})a}$ gauge fields in order to derive a gauge invariant action $\Gamma[A^{(0)}]$. For the $A_\mu^{(0)a}$ gauge fields, we will use the so-called background gauge [45]. To introduce this gauge-fixing procedure, it is convenient to rescale the gauge field as $g A_\mu^{(0)a} \rightarrow A_\mu^{(0)a}$, so that the constant coupling is removed from the covariant derivative and moved into the Yang-Mills Lagrangian, which becomes

$$-\frac{1}{4} F_{\mu\nu}^{(0)a} F_{\mu\nu}^{(0)a} \rightarrow -\frac{1}{4g^2} F_{\mu\nu}^{(0)a} F_{\mu\nu}^{(0)a}, \quad (\text{III.4})$$

where now the covariant derivative and the curvature are given by

$$D_\mu^{(\mathbb{Q})} = \partial_\mu - A_\mu^{(\mathbb{Q})a} t^a, \quad (\text{III.5})$$

$$F_{\mu\nu}^{(\mathbb{Q})a} = \partial_\mu A_\nu^{(\mathbb{Q})a} - \partial_\nu A_\mu^{(\mathbb{Q})a} + f^{abc} A_\mu^{(\mathbb{Q})b} A_\nu^{(\mathbb{Q})c}. \quad (\text{III.6})$$

The gauge field $A_\mu^{(\mathbb{Q})a}$ is split into a classical background field and a fluctuating quantum field:

$$A_\mu^{(\mathbb{Q})a} \rightarrow A_\mu^{(\mathbb{Q})a} + Q_\mu^{(\mathbb{Q})a}, \quad (\text{III.7})$$

where the classical part $A_\mu^{(\mathbb{Q})a}$ is treated as a fixed field configuration and the fluctuating part $Q_\mu^{(\mathbb{Q})a}$ is taken as an integration variable in the functional integral. From Eq.(III.25a) of Part I, one finds that $A_\mu^{(\mathbb{Q})a}$ transforms as a gauge field, whereas $Q_\mu^{(\mathbb{Q})a}$ do it as a matter field in the adjoint representation. This fact can also be seen from the new way of the curvatures, which now become

$$F_{\mu\nu}^{(\mathbb{Q})a} \rightarrow F_{\mu\nu}^{(\mathbb{Q})a} + \mathcal{D}_\mu^{(\mathbb{Q})ab} Q_\nu^{(\mathbb{Q})b} - \mathcal{D}_\nu^{(\mathbb{Q})ab} Q_\mu^{(\mathbb{Q})b} + f^{abc} Q_\mu^{(\mathbb{Q})b} Q_\nu^{(\mathbb{Q})c}. \quad (\text{III.8})$$

To define the functional integral, we introduce a gauge-fixing procedure for the $Q_\mu^{(\mathbb{Q})a}$ fields. We find it convenient to fix the gauge covariantly with respect to the background gauge field, for which we introduce the gauge-fixing functions

$$f^{(\mathbb{Q})a} = \mathcal{D}_\mu^{(\mathbb{Q})ab} Q^{(\mathbb{Q})b\mu}. \quad (\text{III.9})$$

Then, the gauge-fixed standard Lagrangian is given by

$$\begin{aligned} \mathcal{L}_{\text{gauge-fixed}}^{(\mathbb{Q})} = & -\frac{1}{4g^2} \left(F_{\mu\nu}^{(\mathbb{Q})a} + \mathcal{D}_\mu^{(\mathbb{Q})ab} Q_\nu^{(\mathbb{Q})b} - \mathcal{D}_\nu^{(\mathbb{Q})ab} Q_\mu^{(\mathbb{Q})b} + f^{abc} Q_\mu^{(\mathbb{Q})b} Q_\nu^{(\mathbb{Q})c} \right) \\ & \times \left(F_a^{(\mathbb{Q})\mu\nu} + \mathcal{D}^{(\mathbb{Q})a\mu} Q^{(\mathbb{Q})d\nu} - \mathcal{D}^{(\mathbb{Q})a\nu} Q^{(\mathbb{Q})d\mu} + f^{ade} Q^{(\mathbb{Q})d\mu} Q^{(\mathbb{Q})e\nu} \right) \\ & - \frac{1}{2\xi g^2} \left(\mathcal{D}_\mu^{(\mathbb{Q})ab} Q^{(\mathbb{Q})b\mu} \right)^2 + \bar{C}^{(\mathbb{Q})a} \left(\mathcal{D}_\mu^{(\mathbb{Q})ab} \mathcal{D}^{(\mathbb{Q})bc\mu} + f^{abc} \mathcal{D}_\mu^{(\mathbb{Q})bd} Q_\mu^{(\mathbb{Q})d} \right) C^{(\mathbb{Q})c} \\ & + \bar{F}^{(\mathbb{Q})} \left(i\not{D}^{(\mathbb{Q})} + m_{F^{(\mathbb{Q})}} + Q_\mu^{(\mathbb{Q})a} \gamma^\mu t^a \right) F^{(\mathbb{Q})}. \end{aligned} \quad (\text{III.10})$$

Notice that (III.10) is a gauge-fixed Lagrangian with respect to the fluctuating $Q_\mu^{(\mathbb{Q})a}$ fields, but it is invariant under background field gauge transformations, under which the $Q_\mu^{(\mathbb{Q})a}$ transform as matter fields in the adjoint representation.

To compute the standard contribution to the effective action $\Gamma[A^{(\mathbb{Q})}]$, we will need, from Eq. (III.10), those terms which are quadratic in $Q_\mu^{(\mathbb{Q})a}$ and $F^{(\mathbb{Q})}$. Taking the Feynman-'t Hooft gauge and using integration by parts, we obtain

$$\begin{aligned} \mathcal{L}_q^{(\mathbb{Q})} = & \frac{1}{2g^2} Q_\mu^{(\mathbb{Q})a} \left(\mathcal{D}_\rho^{(\mathbb{Q})ab} \mathcal{D}^{(\mathbb{Q})bc\rho} g^{\mu\nu} + 2f^{abc} F^{(\mathbb{Q})b\mu\nu} \right) Q_\nu^{(\mathbb{Q})c} \\ & + \bar{C}^{(\mathbb{Q})a} \left(\mathcal{D}_\mu^{(\mathbb{Q})ab} \mathcal{D}^{(\mathbb{Q})bc\mu} \right) C^{(\mathbb{Q})c} + \bar{F}^{(\mathbb{Q})} \left(i\not{D}^{(\mathbb{Q})} + m_{F^{(\mathbb{Q})}} \right) F^{(\mathbb{Q})}. \end{aligned} \quad (\text{III.11})$$

As far as the gauge-fixing procedure for the KK $A_\mu^{(m)a}$ gauge fields is concerned, we will use the set of gauge-fixing functions

$$f^{(\underline{k})} = \mathcal{D}^{(\mathbb{Q})ab\mu} A_\mu^{(\underline{k})b} - \xi m_{(\underline{k})} A_G^{(\underline{k})a}, \quad (\text{III.12})$$

first introduced in Ref. [46] and which fix the gauge covariantly with respect to $SU(N, \mathcal{M}^4)$. These fixing functions preserve gauge symmetry with respect to the KK zero modes, but they eliminate gauge invariance with respect to the KK excited gauge modes [47]. In this way, invariance of the effective action $\Gamma[A^{(\mathbb{Q})}]$ under the SGTs is assured. From sections III and IV of Part I, one finds, after integration by parts, that the Lagrangian containing the quadratic terms in the KK excitations of gauge and fermions fields is given by:

$$\begin{aligned} \mathcal{L}_q^{\text{KK}} = & \frac{1}{2} \sum_{(\underline{k})} A^{(\underline{k})a\mu} \left[g_{\mu\nu} \left(\mathcal{D}_\rho^{(\mathbb{Q})ab} \mathcal{D}^{(\mathbb{Q})bc\rho} + \delta^{ac} m_{(\underline{k})}^2 \right) - \left(1 - \frac{1}{\xi} \right) \mathcal{D}_\mu^{(\mathbb{Q})ab} \mathcal{D}_\nu^{(\mathbb{Q})bc} + 2g_s f^{abc} A_{\mu\nu}^{(\mathbb{Q})b} \right] A^{(\underline{k})c\nu} \\ & - \frac{1}{2} \sum_{(\underline{k})} A_{\bar{n}}^{(\underline{k})a} \left[\mathcal{D}_\rho^{(\mathbb{Q})ab} \mathcal{D}^{(\mathbb{Q})bc\rho} + \delta^{ac} m_{(\underline{k})}^2 \right] A_{\bar{n}}^{(\underline{k})c} - \frac{1}{2} \sum_{(\underline{k})} A_G^{(\underline{k})a} \left[\mathcal{D}_\rho^{(\mathbb{Q})ab} \mathcal{D}^{(\mathbb{Q})bc\rho} + \delta^{ac} \xi m_{(\underline{k})}^2 \right] A_G^{(\underline{k})c} \\ & + \sum_{(\underline{k})} \bar{C}^{(\underline{k})a} \left[\mathcal{D}_\rho^{(\mathbb{Q})ab} \mathcal{D}^{(\mathbb{Q})bc\rho} + \delta^{ac} \xi m_{(\underline{k})}^2 \right] C^{(\underline{k})c} \\ & + \sum_{(\underline{k})} \bar{F}_V^{(\underline{k})} \left[i\not{D}^{(\mathbb{Q})} - M_{F^{(\underline{k})}} \right] F_V^{(\underline{k})}, \end{aligned} \quad (\text{III.13})$$

where the $F_V^{(\underline{m})}$ fields are defined in Sec. IVB of Part I. In this expression, ξ denotes the gauge-fixing parameter, and $\overline{C}^{(\underline{k})a}$ and $C^{(\underline{k})a}$ are the ghost fields, which arise as part of the quantization procedure [46]. For the sake of simplicity, we shall work, as in the case of the standard theory, in the Feynman-'t Hooft gauge, meaning that $\xi = 1$. This choice simplifies $\mathcal{L}_q^{\text{KK}}$, eliminating the term involving $\mathcal{D}_\mu^{(\underline{Q})}\mathcal{D}_\nu^{(\underline{Q})}$. Moreover, in this gauge, the unphysical masses of pseudo-Goldstone bosons and ghost fields are the same as those of the Kaluza-Klein gauge and scalar fields.

In what follows, we work with the couplings $\mathcal{L}_\beta = \mathcal{L}_q^{(\underline{Q})} + \mathcal{L}_q^{\text{KK}}$. We are interested in integrating out, from \mathcal{L}_β , all the zero-mode gauge and fermion fields together with their KK excited modes, which lead us to define an effective action, $\Gamma[A^{(\underline{Q})}]$, by

$$\begin{aligned} e^{i\Gamma[A^{(\underline{Q})}]} &= \prod_{(\underline{m})} \prod_{x,a,\mu,\bar{n}} \int \mathcal{D}Q_\mu^{(\underline{Q})a} \mathcal{D}F^{(\underline{Q})} \mathcal{D}F_V^{(\underline{m})} \mathcal{D}G_\mu^{(\underline{m})a} \mathcal{D}G_{\bar{n}}^{(\underline{m})a} \mathcal{D}G_G^{(\underline{m})a} \mathcal{D}C^{(\underline{m})a} \exp \left\{ i \int d^4x \mathcal{L}_\beta \right\} \\ &= \exp \left\{ i \int d^4x \left(-\frac{1}{4g^2} F_{\mu\nu}^{(\underline{Q})a} F^{(\underline{Q})a\mu\nu} \right) \right\} (\text{Det } \Delta_{Q^{(\underline{Q})}})^{-1/2} (\text{Det } \Delta_{F^{(\underline{Q})}})^{n_f/2} (\text{Det } \Delta_{C^{(\underline{Q})}})^{+1} \\ &\quad \times \prod_{(\underline{m})} (\text{Det } \Delta_{A^{(\underline{m})}})^{-1/2} (\text{Det } \Delta_{F^{(\underline{m})}})^{n_f/2} (\text{Det } \Delta_{C^{(\underline{m})}})^{+1} (\text{Det } \Delta_{A_{\bar{n}}^{(\underline{m})}})^{-\bar{n}/2} (\text{Det } \Delta_{A_G^{(\underline{m})}})^{-1/2} \quad (\text{III.14}) \end{aligned}$$

Defining the effective Lagrangian $\mathcal{L}_\beta^{\text{eff}}$ by $\Gamma[A^{(\underline{Q})}] = \exp\{i \int d^4x \mathcal{L}_\beta^{\text{eff}}\}$ and performing all the necessary Gaussian integrals, we write down the equation

$$\begin{aligned} \int d^4x \mathcal{L}_\beta^{\text{eff}} &= \int d^4x \left(-\frac{1}{4g^2} F_{\mu\nu}^{(\underline{Q})a} F^{(\underline{Q})a\mu\nu} \right) - i \sum_F \text{Tr} \log \{ i \mathcal{P}^{(\underline{Q})} \otimes \mathbf{1}_f - M_{F^{(\underline{Q})}} \otimes \mathbf{1}_\lambda \otimes \mathbf{1}_D \} \\ &\quad - i 2^{n/2} \sum_{(\underline{k})} \sum_F \text{Tr} \log \{ i \mathcal{P}^{(\underline{Q})} \otimes \mathbf{1}_f - M_{F^{(\underline{k})}} \otimes \mathbf{1}_\lambda \otimes \mathbf{1}_D \} \\ &\quad + \frac{i}{2} \text{Tr} \log \{ g_{\mu\nu} \otimes (-(\mathcal{D}^{(\underline{Q})})^2) + 2i F_{\mu\nu}^{(\underline{Q})a} \otimes T_a^a \} + \frac{i}{2} \text{Tr} \log \{ -(\mathcal{D}^{(\underline{Q})})^2 \} \\ &\quad + \frac{i}{2} \sum_{(\underline{k})} \text{Tr} \log \{ g_{\mu\nu} \otimes (-(\mathcal{D}^{(\underline{Q})})^2 - m_{(\underline{k})}^2 \cdot \mathbf{1}_a) + 2i F_{\mu\nu}^{(\underline{Q})a} \otimes T_a^a \} \\ &\quad + \frac{i}{2} (n-2) \sum_{(\underline{k})} \text{Tr} \log \{ -(\mathcal{D}^{(\underline{Q})})^2 - m_{(\underline{k})}^2 \cdot \mathbf{1}_a \}. \quad (\text{III.15}) \end{aligned}$$

In this expression, the symbol “Tr” denotes a trace over spacetime coordinates and over internal degrees of freedom as well. We have used Kronecker products to emphasize that the arguments of the logarithms are different of each other, living in different spaces. Accordingly, there appear several identity matrices: $\mathbf{1}_f$ is the identity matrix in fermion-flavor space; $\mathbf{1}_\lambda$ is the $N \times N$ identity matrix associated to the fundamental representation of $SU(N, \mathcal{M}^4)$; $\mathbf{1}_D$ is the identity matrix in the space of the Dirac matrices; and $\mathbf{1}_a$ is the identity matrix of size $(N^2 - 1) \times (N^2 - 1)$ that corresponds to the adjoint representation of the $SU(N, \mathcal{M}^4)$ gauge group. Note that the third and fifth traces include the symbol $g_{\mu\nu}$, which should not be understood as a number, but rather as a 4×4 matrix associated to the four-dimensional Lorentz group. In the same sense, $F_{\mu\nu}^{(\underline{Q})a}$ is also a 4×4 matrix. The last trace, involving the factor $n - 2$, is the total contribution from the KK scalars $A_{\bar{n}}^{(\underline{k})a}$, from the pseudo-Goldstone bosons $A_G^{(\underline{k})a}$, and from the ghost fields $C^{(\underline{k})a}$ and $\overline{C}^{(\underline{k})a}$, all summed together.

Consider the Fourier transform, $\bar{A}_\mu^{(\underline{Q})a}(p)$, of the gluon field $A_\mu^{(\underline{Q})a}(x)$, defined by

$$A_\mu^{(\underline{Q})a}(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ip \cdot x} \bar{A}_\mu^{(\underline{Q})a}(p). \quad (\text{III.16})$$

Then the one-loop correction is given by

$$\int d^4x \left(-\frac{1}{4} \mathcal{C} \right) F_{\mu\nu}^{(\underline{Q})a} F^{(\underline{Q})a\mu\nu} = -\frac{1}{2} \int \frac{d^4p}{(2\pi)^4} \bar{A}_\mu^{(\underline{Q})a}(-p) \bar{A}_\nu^{(\underline{Q})a}(p) (p^2 g^{\mu\nu} - p^\mu p^\nu) \mathcal{C} + \dots \quad (\text{III.17})$$

The idea is writing the traces in the effective Lagrangian given in Eq. (III.15) as the last equation and then identify the constant \mathcal{C} . This shall provide us the change of the coupling constant g , as

$$\frac{1}{4g^2} F_{\mu\nu}^{(\underline{Q})a} F^{(\underline{Q})a\mu\nu} \rightarrow \frac{1}{4} \left(\frac{1}{g^2} + \mathcal{C} \right) F_{\mu\nu}^{(\underline{Q})a} F^{(\underline{Q})a\mu\nu}, \quad (\text{III.18})$$

and in turn it will yield the total extra-dimensional contribution to the β function. With such information at hand, asymptotic freedom can be discussed. In order to shorten the expressions that follow, we use the notation $B_1(f(\underline{k})) = B_0(p^2, m_{f(\underline{k})}^2, m_{f(\underline{k})}^2)$, $B_2(f(\underline{k})) = B_0(0, m_{f(\underline{k})}^2, m_{f(\underline{k})}^2)$, with the case $(\underline{k}) = (\underline{0})$ included, and $B_1(\underline{k}) = B_0(p^2, m_{\underline{k}}^2, m_{\underline{k}}^2)$, $B_2(\underline{k}) = B_0(0, m_{\underline{k}}^2, m_{\underline{k}}^2)$, where, strictly, $(\underline{k}) \neq (\underline{0})$.

Our next objective is the calculation of all these traces. We note, from Eq. (III.15), that the generic structure of the traces from quark zero modes is the same as that from the quark excited modes, so that such traces can be solved at once. Using standard methods [47], we find that

$$\begin{aligned} -i \text{Tr} \log \{ i \mathcal{D}^{(\underline{0})} \otimes \mathbf{1}_f - M_{F(\underline{k})} \otimes \mathbf{1}_\lambda \otimes \mathbf{1}_D \} &= -\frac{1}{2} \int \frac{d^4 p}{(2\pi)^4} \bar{A}_\mu^{(\underline{0})a}(-p) \bar{A}_\nu^{(\underline{0})a}(p) (p^2 g^{\mu\nu} - p^\mu p^\nu) \\ &\times \sum_\alpha \frac{1}{(4\pi)^2} \left\{ \frac{2}{3} B_1(\alpha(\underline{k})) - \frac{2}{9} - \frac{4}{3} \frac{m_{\alpha(\underline{k})}^2}{p^2} [B_2(\alpha(\underline{k})) - B_1(\alpha(\underline{k}))] \right\} + \dots, \end{aligned} \quad (\text{III.19})$$

where the Kaluza-Klein combination (\underline{k}) includes the case $(\underline{0})$. In this expression, the sum \sum_α runs over all fermion species. Eq. (III.15) also involves gauge traces, which turn out to be given by

$$\begin{aligned} \frac{i}{2} \text{Tr} \log \{ g_{\mu\nu} \otimes (-(\mathcal{D}^{(\underline{0})})^2) + 2i F_{\mu\nu}^{(\underline{0})a} \otimes T_a^a \} &= -\frac{1}{2} \int \frac{d^4 p}{(2\pi)^4} \bar{A}_\mu^{(\underline{0})a}(-p) \bar{A}_\nu^{(\underline{0})a}(p) (p^2 g^{\mu\nu} - p^\mu p^\nu) \\ &\times \frac{1}{(4\pi)^2} \left\{ -10 B_1(\underline{0}) + \frac{4}{3} \right\} \\ &+ \dots, \end{aligned} \quad (\text{III.20})$$

$$\begin{aligned} \frac{i}{2} \text{Tr} \log \{ g_{\mu\nu} \otimes (-(\mathcal{D}^{(\underline{0})})^2 - m_{(\underline{k})}^2 \cdot \mathbf{1}_a) + 2i F_{\mu\nu}^{(\underline{0})a} \otimes T_a^a \} &= -\frac{1}{2} \int \frac{d^4 p}{(2\pi)^4} \bar{A}_\mu^{(\underline{0})a}(-p) \bar{A}_\nu^{(\underline{0})a}(p) (p^2 g^{\mu\nu} - p^\mu p^\nu) \\ &\times \frac{1}{(4\pi)^2} \left\{ -10 B_1(\underline{k}) + \frac{4}{3} + \frac{8m_{(\underline{k})}^2}{p^2} [B_2(\underline{k}) - B_1(\underline{k})] \right\} \\ &+ \dots, \end{aligned} \quad (\text{III.21})$$

where $B_1(\underline{0}) = B_0(p^2, 0, 0)$ and $(\underline{k}) \neq (\underline{0})$. The fourth trace, which corresponds to the standard ghost fields contribution, can be expressed as

$$\begin{aligned} \frac{i}{2} \text{Tr} \log \{ -(\mathcal{D}^{(\underline{0})})^2 \} &= -\frac{1}{2} \int \frac{d^4 p}{(2\pi)^4} \bar{A}_\mu^{(\underline{0})a}(-p) \bar{A}_\nu^{(\underline{0})a}(p) (p^2 g^{\mu\nu} - p^\mu p^\nu) \\ &\times \frac{1}{(4\pi)^2} \left\{ \frac{1}{2} B_1(\underline{0}) + \frac{1}{3} \right\} + \dots. \end{aligned} \quad (\text{III.22})$$

The last trace, coming from the contributions of physical scalars, pseudo-Goldstone bosons and ghost fields, is expressed as

$$\begin{aligned} \frac{i}{2} \text{Tr} \log \{ -(\mathcal{D}^{(\underline{0})})^2 - m_{(\underline{k})}^2 \cdot \mathbf{1}_a \} &= -\frac{1}{2} \int \frac{d^4 p}{(2\pi)^4} \bar{A}_\mu^{(\underline{0})a}(-p) \bar{A}_\nu^{(\underline{0})a}(p) (p^2 g^{\mu\nu} - p^\mu p^\nu) \\ &\times \frac{1}{(4\pi)^2} \left\{ \frac{1}{2} B_1(\underline{k}) + \frac{1}{3} + \frac{2m_{(\underline{k})}^2}{p^2} [B_2(\underline{k}) - B_1(\underline{k})] \right\} + \dots, \end{aligned} \quad (\text{III.23})$$

provided that $(\underline{k}) \neq (\underline{0})$. Notice that in the (III.22) and (III.23) expressions the factor -2 , corresponding to the ghost

contribution, has already been included. Adding all the contributions together, we get

$$\begin{aligned}
\hat{\mathcal{C}} &= g^2 \mathcal{C} \\
&= \sum_f \frac{g^2}{(4\pi)^2} \frac{2}{3} \left\{ B_1(f^{(\underline{0})}) + 2^{n/2} \sum_{(\underline{m})} B_1(f^{(\underline{m})}) - \frac{1}{3} \left(1 + 2^{n/2} \sum_{(\underline{m})} \right) \right. \\
&\quad \left. - \frac{2}{p^2} \left(m_{f^{(\underline{0})}}^2 [B_2(f^{(\underline{0})}) - B_1(f^{(\underline{0})})] + \sum_{(\underline{m})} m_{f^{(\underline{m})}}^2 [B_2(f^{(\underline{m})}) - B_1(f^{(\underline{m})})] \right) \right\} \\
&\quad + \frac{Ng^2}{(4\pi)^2} \left\{ -\frac{11}{3} B_1(\underline{0}) + \frac{5}{9} + \sum_{(\underline{m})} \left[-\frac{22-n}{6} B_1(\underline{m}) + \frac{2+n}{9} + \frac{2(2+n)}{3} \frac{m_{\underline{m}}^2}{p^2} [B_2(\underline{m}) - B_1(\underline{m})] \right] \right\} \quad (\text{III.24})
\end{aligned}$$

with the sum \sum_f running over all the fermions. In this equation, $(\underline{m}) \neq (\underline{0})$ even in the case of the fermion contributions.

B. Renormalization prescription

In order to calculate the one-loop shift of the charge g , we need to introduce a renormalization prescription at some scale $M^2 = p^2$. We will focus on two ranges of energy; a low-energy region given by $p^2 \ll R^{-2}$ and the very high energy region or massless limit determined by $p^2 \gg R^{-2}$. Here, we focus on the former scenario, deferring the study of the latter one for the next subsection, in which we will analyze the phenomenon of asymptotic freedom.

The result (III.24) can be conveniently written as follows:

$$\begin{aligned}
\hat{\mathcal{C}} &= \frac{g^2}{(4\pi)^2} \left\{ \sum_f \left(\frac{2}{3} \right) \left[\int_0^1 dx \log \left(\frac{\hat{\Lambda}^2}{\Delta_{f^{(\underline{0})}}^2} \right) + 2^{\frac{n}{2}} F_2(c_{2F}^2) + \hat{\mathcal{F}}_2^1(c_{2F}^2) + \mathcal{F}^{(\underline{0})} + \mathcal{F}_n \right] \right. \\
&\quad \left. + N \left[-\frac{11}{3} \int_0^1 dx \log \left(\frac{\hat{\Lambda}^2}{\Delta_{(\underline{0})}^2} \right) - \left(\frac{22-n}{6} \right) F_2(c_{2A}^2) + \hat{\mathcal{F}}_2^1(c_{2A}^2) + \mathcal{B}^{(\underline{0})} + \mathcal{B}_n \right] \right\}, \quad (\text{III.25})
\end{aligned}$$

where the parts containing SDs are:

$$B_1(f^{(\underline{0})}) = \int_0^1 dx \log \left(\frac{\hat{\Lambda}^2}{\Delta_{f^{(\underline{0})}}^2} \right), \quad (\text{III.26a})$$

$$-\frac{11}{3} B_1(\underline{0}) = -\frac{11}{3} \int_0^1 dx \log \left(\frac{\hat{\Lambda}^2}{\Delta_{(\underline{0})}^2} \right), \quad (\text{III.26b})$$

with

$$\Delta_{f^{(\underline{0})}}^2 = m_{f^{(\underline{0})}}^2 - x(1-x)p^2, \quad (\text{III.27a})$$

$$\Delta_{(\underline{0})}^2 = -x(1-x)p^2, \quad (\text{III.27b})$$

$$\hat{\Lambda}^2 = 4\pi e^{-\gamma} e^{\frac{1}{2-\frac{n}{2}}} \mu^2, \quad (\text{III.27c})$$

being μ the scale of dimensional regularization. On the other hand, the combined effects of SDs and NSDs generated by the KK particles are expressed in terms of regularized Epstein functions as follows:

$$F_2(c_{2F}^2) = \sum_{(\underline{m})} B_1(f^{(\underline{m})}), \quad (\text{III.28a})$$

$$F_2(c_{2A}^2) = \sum_{(\underline{m})} B_1(\underline{m}), \quad (\text{III.28b})$$

where the $F_2(c_{2F}^2)$ and $F_2(c_{2A}^2)$ functions are given by Eq.(II.41), with $c_{2F}^2 = \Delta_{f^{(\underline{0})}}^2/R^{-2}$ and $c_{2A}^2 = \Delta_{(\underline{0})}^2/R^{-2}$. The values of these functions, for some numbers of extra dimensions, are listed in Table II. As widely discussed in subsection II C, the Weldon's method to regularize Epstein functions allows us to remove simultaneously both SDs and NSDs,

such as those appearing on the right side of the above equations. However, this method does not allow us to remove nondecoupling effects present in the $F_2(c_{2F}^2)$ and $F_2(c_{2A}^2)$ functions, since they diverge in the $R^{-1} \rightarrow \infty$ limit. This means that we have to choose an appropriate renormalization prescription in order to remove this type of nonphysical effects.

Nondecoupling effects also arise from the following terms:

$$\hat{\mathcal{F}}_2^1(c_{2F}^2) = \frac{2}{p^2} \hat{\mathcal{F}}_2^1(c_{2F(2)}^2, c_{2F(1)}^2), \quad (\text{III.29a})$$

$$\hat{\mathcal{F}}_2^1(c_{2A}^2) = -\frac{2(2+n)}{3p^2} \hat{\mathcal{F}}_2^1(c_{2A(2)}^2, c_{2A(1)}^2), \quad (\text{III.29b})$$

where

$$\begin{aligned} \frac{2}{p^2} \hat{\mathcal{F}}_2^1(c_{2F(1)}^2, c_{2F(2)}^2) &= \sum_{(\underline{m})} \frac{2m_{f(\underline{m})}^2}{p^2} [B_1(f(\underline{m})) - B_2(f(\underline{m}))] \\ &= \frac{2}{p^2} [\hat{\mathcal{F}}_2^1(c_{2F(1)}^2) - \hat{\mathcal{F}}_2^1(c_{2F(2)}^2)] \end{aligned} \quad (\text{III.30a})$$

$$\begin{aligned} -\frac{2(2+n)}{3p^2} \hat{\mathcal{F}}_2^1(c_{2A(1)}^2, c_{2A(2)}^2) &= -\frac{2(2+n)}{3} \sum_{(\underline{m})} \frac{m_{\underline{m}}^2}{p^2} [B_1(\underline{m}) - B_2(\underline{m})] \\ &= -\frac{2(2+n)}{3p^2} [\hat{\mathcal{F}}_2^1(c_{2A(1)}^2) - \hat{\mathcal{F}}_2^1(c_{2A(2)}^2)], \end{aligned} \quad (\text{III.30b})$$

with the $\hat{\mathcal{F}}_2^1(c_{2F(1)}^2)$, $\hat{\mathcal{F}}_2^1(c_{2F(2)}^2)$, $\hat{\mathcal{F}}_2^1(c_{2A(1)}^2)$, and $\hat{\mathcal{F}}_2^1(c_{2A(2)}^2)$ functions given by Eq.(II.50). Although in this case the $\hat{\mathcal{F}}_2^1(c_{2F(2)}^2, c_{2F(1)}^2)$ and $\hat{\mathcal{F}}_2^1(c_{2A(2)}^2, c_{2A(1)}^2)$ functions do not diverge in the $R^{-1} \rightarrow \infty$ limit, they tend to a finite nonzero value, so the renormalization prescription chosen must also remove these nondecoupling effects, since they are unacceptable in a physical amplitude. It is worth looking more closely at the expressions given by Eqs.(III.30a) and (III.30b). First, note that $c_{2F(1)}^2 = c_{2F}^2$, $c_{2F(2)}^2 = c_0^2 = m_{f(\mathbb{Q})}^2/R^{-2}$, $c_{2A(1)}^2 = c_{2A}^2$, and $c_{2A(2)}^2 = 0$. Then, after some straightforward algebraic manipulations, one obtains

$$\frac{2}{p^2} \hat{\mathcal{F}}_2^1(c_{2F}^2, c_0^2) = \frac{1}{3} (\alpha_2^n - \beta_2^n) + \beta_2^n f(\tau) + O\left(\frac{1}{R^{-2}}\right), \quad (\text{III.31})$$

where

$$f(\tau) = \tau \left[-1 + \sqrt{\tau-1} \arctan\left(\frac{1}{\sqrt{\tau-1}}\right) \right], \quad \tau = \frac{4m_{f(\mathbb{Q})}^2}{p^2}. \quad (\text{III.32})$$

On the other hand,

$$\hat{\mathcal{F}}_2^1(c_{2A}^2) = -R^{-2} I_2 \left[\alpha_1^n + (\alpha_2^n + \gamma_1^n) c_{2A}^2 + O\left(\frac{1}{R^{-4}}\right) \right], \quad (\text{III.33a})$$

$$\hat{\mathcal{F}}_2^1(0) = -R^{-2} I_2 [\alpha_1^n], \quad (\text{III.33b})$$

where the relation given by Eq.(III.33b) is exact. Therefore,

$$-\frac{2(2+n)}{3p^2} \hat{\mathcal{F}}_2^1(c_{2A}^2, 0) = -\frac{2(2+n)}{12} (\alpha_2^n + \gamma_1^n) + O\left(\frac{1}{R^{-2}}\right). \quad (\text{III.34})$$

Finally,

$$\begin{aligned} \mathcal{F}^{(\mathbb{Q})} &= -\frac{1}{2} - \frac{2m_{f(\mathbb{Q})}^2}{p^2} [B_2(f(\mathbb{Q})) - B_1(f(\mathbb{Q}))] \\ &= -\frac{1}{2} - \frac{2m_{f(\mathbb{Q})}^2}{p^2} \int_0^1 dx \log\left(\frac{\Delta_{(\mathbb{Q})}^2}{m_{f(\mathbb{Q})}^2}\right), \end{aligned} \quad (\text{III.35a})$$

$$\mathcal{B}^{(\mathbb{Q})} = \frac{5}{9}, \quad (\text{III.35b})$$

$$\begin{aligned}\mathcal{F}_n &= -2^{\frac{n}{2}-1} \sum_{(m)} \\ &= \frac{2^n - 1}{2^{\frac{n}{2}+1}},\end{aligned}\tag{III.36a}$$

$$\begin{aligned}\mathcal{B}_n &= \left(\frac{2+n}{9}\right) \sum_{(m)} \\ &= -\left(\frac{2+n}{9}\right) \left(\frac{2^n - 1}{2^n}\right).\end{aligned}\tag{III.36b}$$

Note that \mathcal{F}_n and \mathcal{B}_n vanish for $n = 0$, which means that these effects are of decoupling nature.

In the standard theory, the counterterm δ_A removes the SDs appearing in the expressions (III.26a) and (III.26b), but in our case we also require it to remove the nondecoupling effects given by the terms (III.30a), (III.30b), (III.31), and (III.34). Our renormalization prescription at a scale $M < R^{-1}$ contains the following ingredients: (1) the SDs are removed through the $\overline{\text{MS}}$ renormalization scheme and (2) a finite term is introduced to remove nondecoupling effects, so the renormalized action reduces to the standard one in the $R^{-1} \rightarrow \infty$ limit. The fulfillment of these requirements implies then that the counterterm is given by

$$\begin{aligned}\delta_A &= -\frac{g^2}{(4\pi)^2} \left\{ \sum_f \left(\frac{2}{3}\right) \left[\int_0^1 dx \log \left(\frac{\hat{\Delta}_{f(\mathbb{Q})}^2}{\Delta_{f(\mathbb{Q})}^2} \right) + 2^{\frac{n}{2}} F(\bar{c}_{2F}^2) + \hat{\mathcal{F}}_2^1(\bar{c}_{2F}^2) \right] \right. \\ &\quad \left. + N \left[-\frac{11}{3} \int_0^1 dx \log \left(\frac{\hat{\Delta}_{(\mathbb{Q})}^2}{\Delta_{(\mathbb{Q})}^2} \right) - \frac{22-n}{6} F(\bar{c}_{2A}^2) + \hat{\mathcal{F}}_2^1(\bar{c}_{2A}^2) \right] \right\},\end{aligned}\tag{III.37}$$

where

$$F(\bar{c}_{2F}^2) = -\alpha_2^n - \beta_2^n \int_0^1 dx \log(\bar{c}_{2F}^2),\tag{III.38a}$$

$$F(\bar{c}_{2A}^2) = -\alpha_2^n - \beta_2^n \int_0^1 dx \log(\bar{c}_{2A}^2).\tag{III.38b}$$

$$\hat{\mathcal{F}}_2^1(\bar{c}_{2F}^2) = -\frac{1}{3}(\alpha_2^n - \beta_2^n) - \beta_2^n f(\tau),\tag{III.39a}$$

$$\hat{\mathcal{F}}_2^1(\bar{c}_{2A}^2) = \frac{2(2+n)}{12}(\alpha_2^n + \gamma_1^n).\tag{III.39b}$$

In the above expressions, the bar over the diverse Δ s and c^2 s functions means that p^2 has been replaced by M^2 , that is,

$$\bar{\Delta}_{f(\mathbb{Q})}^2 = m_{f(\mathbb{Q})}^2 - x(1-x)M^2,\tag{III.40a}$$

$$\bar{\Delta}_{(\mathbb{Q})}^2 = -x(1-x)M^2.\tag{III.40b}$$

Then, the original coupling constant in the effective action is replaced by the running coupling constant

$$g^2(p^2) = \frac{g^2}{1 + \bar{\mathcal{C}} + \delta_A},\tag{III.41}$$

where

$$\begin{aligned}\bar{\mathcal{C}} + \delta_A &= \frac{g^2}{(4\pi)^2} \left\{ \int_0^1 dx \left[\sum_f \left(\frac{2}{3}\right) \log \left(\frac{\bar{\Delta}_{f(\mathbb{Q})}^2}{\Delta_{f(\mathbb{Q})}^2} \right) - \frac{11N}{3} \log \left(\frac{\bar{\Delta}_{(\mathbb{Q})}^2}{\Delta_{(\mathbb{Q})}^2} \right) \right] \right. \\ &\quad \left. + \sum_f \left(\frac{2}{3}\right) \left[2^{\frac{n}{2}} [F_2(c_{2F}^2) - F_2(\bar{c}_{2F}^2)] + \hat{\mathcal{F}}_2^1(c_{2F}^2) - \hat{\mathcal{F}}_2^1(\bar{c}_{2F}^2) + \mathcal{F} \right] \right. \\ &\quad \left. + N \left[-\frac{22-n}{6} [F_2(c_{2A}^2) - F_2(\bar{c}_{2A}^2)] + \hat{\mathcal{F}}_2^1(c_{2A}^2) - \hat{\mathcal{F}}_2^1(\bar{c}_{2A}^2) + \mathcal{B} \right] \right\}.\end{aligned}\tag{III.42}$$

This expression is free of nondecoupling effects in the KK heavy mass limit, since

$$\lim_{R^{-1} \rightarrow \infty} [F_2(c_{2F}^2) - F_2(\bar{c}_{2F}^2)] = 0, \quad (\text{III.43a})$$

$$\lim_{R^{-1} \rightarrow \infty} [F_2(c_{2A}^2) - F_2(\bar{c}_{2A}^2)] = 0, \quad (\text{III.43b})$$

$$\lim_{R^{-1} \rightarrow \infty} [\hat{\mathcal{F}}_2^1(c_{2F}^2) - \hat{\mathcal{F}}_2^1(\bar{c}_{2F}^2)] = 0, \quad (\text{III.43c})$$

$$\lim_{R^{-1} \rightarrow \infty} [\hat{\mathcal{F}}_2^1(c_{2A}^2) - \hat{\mathcal{F}}_2^1(\bar{c}_{2A}^2)] = 0. \quad (\text{III.43d})$$

C. The β function

So far, we have presented results at relative low energies, $p^2 < R^{-2}$. It is time to study short distance effects through the β function. In order to investigate the impact of extra dimensions on the β function, we will work in the zero-mass limit, that is, we will assume that $\{m_{f(\underline{m})}, m_{(\underline{m})}\} \ll M$. In this massless scenario, Epstein functions do not appear, so Weldon's method is not applicable and ultraviolet divergences from KK particles emerge explicitly. This means that the set of all the ultraviolet divergences, that is, those induced by the SM model fields and their infinite number of KK excitations, must be removed by the counterterm that renormalizes the standard action. Then, the counterterm becomes

$$\delta_A = -\frac{g^2}{(4\pi)^2} \left\{ \sum_f \left(\frac{2}{3} \right) - N \left(\frac{11}{3} \right) + \sum_{(\underline{m})} \left[2^{\frac{n}{2}} \sum_f \left(\frac{2}{3} \right) - N \left(\frac{11 - \frac{n}{2}}{3} \right) \right] \right\} \int_0^1 dx \log \left(\frac{\hat{\Lambda}^2}{\Delta_{(\underline{Q})}^2} \right). \quad (\text{III.44})$$

At this stage, we use the result given by Eq.(II.39) to express the above expression as follows:

$$\delta_A = -\frac{g^2}{(4\pi)^2} \left\{ \sum_f \left(\frac{2}{3} \right) - N \left(\frac{11}{3} \right) + \left(\frac{1 - 2^n}{2^n} \right) \left[2^{\frac{n}{2}} \sum_f \left(\frac{2}{3} \right) - N \left(\frac{11 - \frac{n}{2}}{3} \right) \right] \right\} \int_0^1 dx \log \left(\frac{\hat{\Lambda}^2}{\Delta_{(\underline{Q})}^2} \right). \quad (\text{III.45})$$

On the other hand, the β function is defined in terms of the counterterm as follows:

$$\beta(g) = gM^2 \frac{\partial \delta_A}{\partial M^2}, \quad (\text{III.46})$$

which leads to

$$\beta(g) = -\frac{g^3}{(4\pi)^2} \left\{ \frac{11}{3}N - \frac{2}{3}n_f + \left(\frac{1 - 2^n}{2^n} \right) \left[\frac{11 - (\frac{n}{2})}{3}N - 2^{\frac{n}{2}} \left(\frac{2}{3} \right) n_f \right] \right\}, \quad (\text{III.47})$$

where n_f represents the number of fermion species, each given in the fundamental representation of $SU(N)$. In the above expression, the first two terms correspond to the well-known result of the usual theory, whereas the terms characterized by Fourier sums represent the contribution from extra dimensions. Several comments, concerning this result, are in order. In first place, it should be noted that with respect to the contribution from the extra dimensions, only short distance effects emerge, as it should be. To appreciate the importance of this fact, let us reanalyze the SDs induced by the KK gauge bosons $A_\mu^{(\underline{m})a}$, which are characterized by the $B_1(\underline{m})$ divergent scalar function. Apart from a numerical factor, this contribution is given by

$$\sum_{(\underline{m})} B_1(\underline{m}) = \begin{cases} \sum_{l=1}^n \binom{n}{l} E_l^{c_{2A}^2} \left(2 - \frac{D}{2} \right) \Gamma \left(2 - \frac{D}{2} \right), & m_{(\underline{m})} \neq 0, \\ \left(\Delta_{(\underline{Q})}^2 \right)^{-(2 - \frac{D}{2})} \Gamma \left(2 - \frac{D}{2} \right) \left(\sum_{(\underline{m})} 1 \right), & m_{(\underline{m})} = 0. \end{cases} \quad (\text{III.48})$$

In the case of an infinite number of KK particles with masses $m_{(\underline{m})} \neq 0$, the regularized Epstein function conspire with the Gamma function to produce a result free of ultraviolet divergences. However, as it can be appreciated from the above expression, this cancellation effect is no longer present when instead an infinite number of massless particles is considered. In the former case, we have a set of nontrivial regularized Epstein functions $E_l^{c^2}(2 - D/2)$ which conspire

with the Gamma function of dimensional regularization to produce a finite result; whereas in the latter case only the Riemann $\zeta(0)$ function appears, which is not affected by the $D \rightarrow 4$ limit.

Returning to the result (III.47), we can identify the contributions from the KK $A_\mu^{(\underline{m})a}$ gauge fields, each given by factor of $(11/3)N$; the contribution from n sets of KK real scalar fields given in the adjoint representation of the group (recall that there are $n - 1$ physical scalars $A_{\bar{n}}^{(\underline{m})a}$, with $\bar{n} = 1, \dots, n - 1$, including the longitudinal component of $A_\mu^{(\underline{m})a}$, characterized by the pseudo-Goldstone boson $A_G^{(\underline{m})a}$), each representation contributing with a factor of $(1/2)N$, as they are real fields. Also, each species of fermions, given in the fundamental representation, contributes with a factor of $(2/3)2^{\frac{n}{2}}$ (recall that each KK excitation of a fermionic particle is characterized by a set of $2^{\frac{n}{2}}$ spinors in addition to the zero mode). Note that all these contributions have the correct sign: gauge fields contribute positively, whereas matter fields do it negatively. If for a moment we put aside the presence of the factor $\sum_{(\underline{m})}$ multiplying these contributions, we can see that the matter contribution would dominate over the gauge contribution for a sufficiently large number of extra dimensions n mainly due to the presence of the factor $2^{\frac{n}{2}}$ in the fermion part. However, the situation changes when the result (II.39), for the multiple sums $\sum_{(\underline{m})}$, is taken into account.

Finally, the running coupling constant (III.41) becomes

$$g^2(p^2) = \frac{g^2}{1 + \frac{g^2}{(4\pi)^2} \left\{ \frac{11}{3}N - \frac{2}{3}n_f + \left(\frac{1-2^n}{2^n}\right) \left[\frac{11 - (\frac{n}{2})}{3}N - 2^{\frac{n}{2}} \left(\frac{2}{3}\right) n_f \right] \right\} \log\left(\frac{p^2}{M^2}\right)}. \quad (\text{III.49})$$

The expression (III.47) shows us that $\beta < 0$ always, so the phenomenon of asymptotic freedom is maintained in presence of extra dimensions. Let us to stress the fact that the above results depend on the dimension of the compact manifold but not on its size, as expected, since the β function only reflects the short distances behavior of a theory. It is very important to note that this result for the β function reduces to that of the standard theory for $n = 0$, which is consistent with the fact that the new physics effects are of decoupling nature.

IV. SUMMARY

The divergent structure of the effective theory that emerges from the compactification of the theory that comprises, besides the $(4 + n)$ -dimensional version of the SM, all the interactions of higher canonical dimensions that are compatible with the extended symmetries, was discussed at the one-loop level from the perspective of renormalization in a modern sense. To do this, the generic one-loop contributions to standard Green's functions (SGFs) were considered. We emphasized that such contributions can lead to two types of divergences, namely, the divergences associated with short distance effects or ultraviolet divergences, which we called standard divergences (SD), and nonstandard divergences (NSD) which may arise as a consequence of adding the contributions from an infinite number of particles (the KK contributions). The virtual contributions from the infinite number of KK particles are given by Epstein functions, which naturally emerge from the implemented compactification scheme, which geometrically recreates the Casimir's effect. To deal with this new kind of divergences, a regularization scheme based on the Epstein zeta function, which is intimately linked to the dimensional regularization approach, was introduced. We would like to stress the fact that both regularization schemes are linked through products of Epstein functions and Gamma functions of the way $E_l^{c_N}(N - D/2)\Gamma(N - D/2)$, so both types de functions naturally emerge defined on the complex plane, as in the dimensional regularization scheme a complex dimension D is required, which allowed us to exploit the interesting features of Epstein functions. In particular, the counterintuitive result $\sum_{(\underline{m})} = (1 - 2^n)/2^n$ played a central role in our study of asymptotic freedom.

The one-loop structure of the SGFs was discussed in a general context and, in particular, in the environment of a non-Abelian gauge theory with fermions in the fundamental representation of the $SU(N)$ group. In the general context of the EDSM, we discussed the divergent structure of this type of Green's functions. In order to tackle the problem of the divergences generated by the infinite number of KK particles, we introduced a regularized (finite) version of the Epstein function, which allows us to remove simultaneously both SDs and NSDs induced by this type of particles. The possibility of renormalizing this type of divergences via a counterterm in the effective Lagrangian was also discussed. However, we decided to use, instead of an explicit counterterm, regularized Epstein functions, since this method offers many advantages in practice. In particular, the removal of SDs is interesting, as it emerges as a limit process in the product $E_l^{c_N}(N - D/2)\Gamma(N - D/2)$ when $D \rightarrow 4$. This limit exists because the poles of the Gamma function, $N - D/2 = 0, -1, \dots$, are the trivial zeros of the regularized Epstein function. Also, the fact that this method does not remove nondecoupling effects is stressed. It was shown how to remove this type of

nonphysical effects through a modification of the counterterm that is necessary to remove the SDs associated with the SM particles. A general outline of how to implement a renormalization prescription, and thus to renormalize physical quantities in a modern sense, was presented. Two possible renormalization prescriptions were studied in the context of Yang-Mills theory with fermions in the fundamental representation, namely, one at a low-energy scale $M \ll R^{-1}$, and another at the short-distance regime, $M \gg R^{-1}$, or massless limit. Both low energy and high energy prescriptions were implemented to calculate the shift of the charge g in this context, including the phenomenon of asymptotic freedom.

Our results concerning the one-loop structure of the EDSM can be summarized as follows: both SDs and NSDs induced by KK particles can be removed simultaneously from the theory if regularized Epstein functions are used. This method only removes the poles of the Epstein function, so nondecoupling effects must be removed through a finite counterterm. Ultraviolet divergences emerge only from SM particles, which can be generated by couplings with renormalizable and nonrenormalizable structures in the Dyson's sense. Our conclusion is that the EDSM is renormalizable in a broader or modern sense. The nondecoupling effects induced by the KK particles can be removed from physical amplitudes by adding a constant counterterm to the counterterm used to renormalize the theory in this sense.

As an application of our approach to extra dimensions, we have studied the one-loop structure of Yang-Mills theories coupled to fermions given in the fundamental representation of the $SU(N, \mathcal{M}^4)$ group. The structure of the diverse interactions is identical to that given for the color group within the context of the EDSM. We have derived an expression for the shift in the coupling constant g by calculating the effective action at the one-loop level. In the bosonic sector, the contributions from the standard gauge fields, $A_\mu^{(0)a}$, and their KK excitations, $A_\mu^{(\underline{m})a}$, including their associated physical scalar KK fields, $A_n^{(\underline{m})a}$, were considered. In the fermionic sector, the contributions from the $\{f^{(0)}, f_{(1)}^{(\underline{m})}, \dots, f_{(2^{\frac{n}{2}})}^{(\underline{m})}\}$ family associated to each fermion species, were considered. A background field gauge was used to maintain invariance of the classical $A_\mu^{(0)a}$ fields under SGTs, whereas a scheme also covariant under SGTs was used to fix the gauge with respect to the nonstandard gauge transformations of the $A_\mu^{(\underline{m})a}$ fields. Besides simplifying considerably the calculations, these gauge-fixing procedures allowed us to derive a one-loop effective action for the Yang-Mills sector that is manifestly invariant under SGTs. Our purpose in this study has been twofold. On one side, our interest was focused on the one-loop structure of the theory and its renormalization, and on the other side, our attention was focused in the phenomenon of asymptotic freedom given its physical relevance in the context of the strong interaction. Our main results in this part of the paper are the following:

- In order to appreciate more clearly the divergent structure of the one-loop contribution to the effective action, we performed the calculations in an exact way, that is, neither standard fermion masses, $m_{f^{(0)}}$, nor Kaluza-Klein masses $m_{(\underline{m})}$ were neglected. Both bosonic and fermionic contributions were naturally organized into four parts: (1) a term characterizing the contribution from the standard fields ($A_\mu^{(0)a}$ or $f^{(0)}$), which has SDs; (2) a term that contains the SDs induced by the infinite number of KK excitations, which, due to the properties of the regularized Epstein functions, produce a finite result, that, however, does not decouple in the Kaluza-Klein heavy mass limit; (3) a term that is free of SDs but that contains infinite sums on Kaluza-Klein amplitudes, which do not decouple in the heavy mass limit; and (4) a term that depends on KK particles only through the infinite sums $\sum_{(\underline{m})} (\text{const.})$, which do not diverge due to the analytical continuation of the Epstein function.

- First, we implement a renormalization prescription at low energies, given by a scale $M^2 = p^2 \ll R^{-2}$. There are no NSDs due to the implementation of regularized Epstein functions, so only the SDs generated by the SM fields and the nondecoupling effects induced by the KK particles must be removed. The counterterm so introduced contains the $\overline{\text{MS}}$ scheme that removes the SDs in (1) and a counter effect that removes the nondecoupling effect in (2).

- To investigate the impact of extra dimensions on the phenomenon of asymptotic freedom, we have worked in the zero mass limit, that is, it was assumed that $\{m_{f^{(0)}}, m_{(\underline{m})}\} \ll M$. In this scenario, the method to remove the divergences induced by the KK particles, which is based on regularized Epstein functions, is not applicable because there are no Epstein functions. However, in this scenario of an infinite number of massless particles, there are no NSDs, as the sum $\sum_{(\underline{m})}$ is indeed a set of nested $\zeta(0) = -1/2$ Riemann functions, which leads to the finite result $\sum_{(\underline{m})} = (1 - 2^n)/2^n$. Ultraviolet divergences also arise, but they can be removed from the theory by the same counterterm that removes the ultraviolet divergences induced by the SM particles, that is, SDs appearing in terms (1) and (2) mentioned above are renormalized by the same $\overline{\text{MS}}$ scheme. Because there is an infinite number of KK particles, the counterterm would contain an infinite sum of terms proportional to $\Gamma(2 - D/2)$, $D \rightarrow 4$. However, due to the analyticity of the Riemann function, such a sum is indeed finite, that is, $\sum_{(\underline{m})} \Gamma(2 - D/2) = [(1 - 2^n)/2^n] \Gamma(2 - D/2)$. As a nice

consequence, the β function only receives contributions from the short distance effects of both standard particles and their KK excitations. It was found that each vector $A_\mu^{(\underline{m})a}$ field contributes to the β function with a factor $(11/3)N$ and that each real scalar $A_n^{(\underline{m})a}$ field do it with a factor $(\frac{1}{2}/3)N$, of which there are a total of n fields (the longitudinal component of the gauge $A_\mu^{(\underline{m})a}$ field contribute as a real scalar field). On the other hand, KK excitations of each specie of fermion, in the fundamental representation, contributes with a factor of $2/3$. However, because of each species is characterized, besides the standard part, by $2^{\frac{n}{2}}$ spinors, the net contribution of the KK excitations is given by $(2/3)2^{\frac{n}{2}}$. Both bosonic and fermionic contributions to the β function have the correct sign, that is, the gauge fields $A_\mu^{(\underline{m})a}$ contribute positively, whereas scalar and fermionic fields contribute negatively, as they are matter fields. All the contributions coming from the KK excitations are proportional to the factor $\sum_{(\underline{m})} = (1/2^n - 1) < 0$, so that the net effect of extra dimensions on the β function has opposite sign to the usual contribution. Due to the presence of the $2^{\frac{n}{2}}$ factor multiplying the fermion contribution, the presence of the $\sum_{(\underline{m})}$ leads to a $\beta < 0$ always. We stress the fact that our result for the β function reduces to the standard value for $n = 0$, which shows the decoupling nature of this class of new physics effects.

We consider worth mentioning the central role played by the Epstein zeta functions used to deal with the infinite sums appearing in loop amplitudes. In particular, the finite result for the $\sum_{(\underline{m})}$ sums has been fundamental in obtaining an unambiguous prediction for the β function. The obtained result shows that the presence of extra dimensions tends to reinforce the phenomenon of asymptotic freedom.

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